A STUDY ON CONVERGENCE THEOREM FOR SOLVING EQUILIBRIUM PROBLEM, FIXED POINT PROBLEM AND VARIATIONAL INEQUALITY PROBLEM VIA RESEARCH OF S.CHANG ET AL.

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ชื่อเรื่อง	การศึกษาทฤษฎีบทการลู่เข้าเพื่อแก้ปัญหาเชิงดุลยภาพ ปัญหาจุดตรึง และปัญหาอสมการการแปรผันผ่านงานวิจัยของ S.Chang และคณะ
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บทคัดย่อ

ในการศึกษาอิสระนี้ เราได้ทำการศึกษาการพิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้มของวิธีการประมาณค่าแบบหนืดเพื่อหาผลเฉลยร่วมของปัญหาจุดตรึงสำหรับวงศ์ที่ไม่จำกัดของการส่งแบบไม่ขยาย ปัญหาเชิงดุลยภาพและปัญหาอสมการการแปรผันสำหรับการส่งทางเดียวอย่างเข้มผกผันในปริภูมิฮิลเบิร์ต นอกจากนี้เราได้ทำการหาตัวอย่างที่สอดคล้องกับทฤษฎีบทหลักที่ได้ศึกษาเพื่อสนับสนุนทฤษฎีบทดังกล่าว

Title	A study on convergence theorem for solving equilibrium problem fixed point problem and variational inequality problem via research of S.chang et al.
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ABSTRACT

In this independent we study the proof of the strong convergence by using the viscosity approximation method for finding a common element of the set of fixed points for a family of infinitely nonexpansive mappings, the set of solution of an equilibrium problem and the set of solution of the variational inequality for an inverse-strongly monotone mapping in a Hilbert space. Furthermore, we propose some examples which corresponding to the main theorem.

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CHAPTER I

Introduction

Let H be a real Hilbert space with inner product $< \cdot, \cdot >$ and norm $|| \cdot ||$, respectively , C be a nonempty closed convex subset of H and P_C be the metric projection of H onto C. In the following , we denote by " $\rightarrow$ " a strong convergence and by " $\rightharpoonup$ " a weak convergence. Recall that a mapping S: $C \rightarrow C$ is called nonexpansive , if

$$\|Sx - Sy\| \leq \|x - y\|. \tag{1.1.1}$$

The set of fixed points of the mapping S is denoted by *F(S)*. Recall that a mapping A: $C \rightarrow H$ is called an α - inverse-strongly monotone[2] , if there exists a positive real number α such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha ||Ax - Ay||^2, \forall x, y \in C. \tag{1.1.2}$$

Remark It is easy to see that if $A : C \rightarrow H$ is an α - inverse-strongly monotone, then it is a $\frac{1}{\alpha}$ - Lipschitzian mapping.
Let A: $C \rightarrow H$ be a mapping the classical variational inequality problem is to find $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \forall v \in C. \tag{1.1.3}$$

The set of solutions of variational inequality (1.1.3) is denoted by *VI(C,A)*. Let $\phi : C \times C \rightarrow \mathbb{R}$ be a bifunction for the function ϕ is to find a point $x_* \in C$ such that

$$\phi(x_*, y) \geq 0, \forall y \in C. \tag{1.1.4}$$

We denote the set of solution of the equiliibrium problem (1.1.4) by $EP(\phi)$.

Literature review

In 1977 Combettes and Hirstoaga [5] introduced an iterative scheme of finding the best approximation to initial data when $EP(\phi)$ is nonempty and proved some strong convergence theorem in Hilbert spaces.

In 2003, Takahashi and Totoda [14] proposed the iterative scheme for finding a common element of $F(S) \cap VI(C, A)$ as shown in the following : $x_1 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n A x_n), n \geq 1 \tag{1.1.5}$$

and obtained a weak convergence theorem in the framework of Hilbert space, where $\{\alpha_n\}$ is a sequence in (0,1) and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$.

For finding a common element of $F(S) \cap EP(\phi)$, Takahashi and Takahashi [13] introduced the following iterative scheme by the viscosity approximation method in a Hilbert space : $x_1 \in H$ and

$$\begin{cases} \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S u_n, \forall n \geq 1. \end{cases} \tag{1.1.6}$$

Recently, for finding a common element of $F(S) \cap VI(C, A) \cap EP(\phi)$, Su, Shang and Qin [11] introduced the following iterative scheme : $x_1 \in H$

$$\begin{cases} \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) SP_C(u_n - \lambda_n A u_n), \forall n \geq 1. \end{cases} \tag{1.1.7}$$

Under suitable conditions some strong convergence theorems are proved which extend and improve the results of Iiduka et al.[7] and Takahashi et al.[13].

On the other hand, in order to find a common element of $F(S)\cap VI(C,A)\cap EP(\phi)$,very recently Plubtieng and Punpaeng [8] also introduced the following iterative scheme : $x_1 = u \in C$ and

$$\begin{cases} \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda_n A y_n). \end{cases} \tag{1.1.8}$$

Under suitable conditions some strong convergence theorems are proved which extend some recent results of Yao and Yao [15].

CHAPTER II

Preliminaries and lemmas

2.1 Preliminaries

In this section we recall some basic definitions which will be useful for the next Chapter.

Definition 2.1.1 Hilbert space

Let H be an inner product space. Then H is called a Hilbert space if for each bounded sequence $\{x_n\}$ of H , there exists a weakly convergent subsequence of $\{x_n\}$.

Definition 2.1.2 Closed set

Let H be a Hilbert space. A subset C of H is called a closed set if $\{x_n\} \subset C$ and $x_n \to x$ imply $x \in C$.

Definition 2.1.3 Convex set

Let C be a subset of a Hilbert space H and scalar $t \in$ (0,1) then C is said to be convex if $tx + (1-t)y \in C$ for all $x, y \in C$.

Definition 2.1.4 Nonexpansive mapping

The mapping $T : C \to C$ is said to be nonexpansive mapping if $||Tx - Ty|| \leq ||x - y||$ for all $x, y \in C$.

Definition 2.1.5 Contraction

Let C be a subset of Hilbert space. A mapping f: $C \to C$ is called a contraction on C if there is a positive real number $\alpha < 1$ such that for all $x, y \in C$,
$||f(x) - f(y)|| \leq \alpha \; ||x - y||$, $\forall x, y \in C$.

Definition 2.1.6 Bounded sequence

A sequence $\{x_n\}$ in H is said to be bounded if there is $M > 0$ such that $||x_n|| \leq M$ for all $n \in N$.

Definition 2.1.7 Strong convergence

A sequence $\{x_n\}$ in Hilbert space H is said to be strong convergence (or convergence in the norm) if there is an $x \in H$ such that $\lim_{n\to\infty} ||x_n - x|| = 0$.

Definition 2.1.8 Fixed point

The point x is a fixed point of the mapping T if $Tx = x$.

Definition 2.1.9 Weakly lower semi continuous

If $(x, y_n) \rightharpoonup (x, y_0)$ (weakly), then $\phi(x, y_0) \leq \liminf_{n\to\infty}(x, y_n)$.

Definition 2.1.10 : **limit superior**

limit superior of x_n is defined by $\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \sup_{m\geq n}(x_m)$.

2.2 Lemmas

Let H be a real Hilbert space. It is well known that for any $\lambda \in [0, 1]$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2.$$

Let C be a nonempty closed convex subset of H, for each $x \in H$. there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \forall y \in C.$$

P is called a metric projection of H onto C. It is known that P_C is a nonexpansive mapping and satisfies :

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \forall x, y \in C. \tag{2.2.1}$$

Moreover, $P_C x$ is characterized by the following property :

$$P_C x \in C\, and\, \langle x - P_C x, P_C x - y\rangle \geq 0, \forall y \in C. \tag{2.2.2}$$

In the context of the variational inequality problem, this implies that

$$u \in VI(C,A) \Leftrightarrow u = P_C(u - \lambda Au), \forall \lambda > 0. \tag{2.2.3}$$

A Banach space X is said to satisfy the Opial condition if for each sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$,we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \forall y \in X, y \neq x.$$

It is well known that each Hilbert space satisfies the Opial condition. A set-valued mapping $T : H \to 2^H$ is said to be monotone, if for all $x, y \in H, f \in Tx$, and $g \in Ty$ imply that$\langle f - g, x - y\rangle \geq 0$.

A monotone mapping $T : H \longrightarrow H$ is said to be maximal [9] , if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping is maximal, if and only if for $(x, f) \in H \times H, \langle f - g, x - y\rangle \geq 0, \forall (y, g) \in G(T)$ imply that $f \in Tx.Let\, A : C \longrightarrow H$ be an inverse-strongly monotone mapping and let $N_C v$ be the normal cone to C at $v \in C$,i.e.,

$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$,

and define

$$Tv = \begin{cases} Av + N_C v, v \in C \\ \emptyset, v \in C. \end{cases} \tag{2.2.4}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C,A)$ (see,for example, [7]). For solving the equilibrium problem for bifunction $\phi : C \times C \longrightarrow \mathbb{R}$, let us assume that ϕ satisfies the following conditions :

$(A1)$ $\phi(x, x) = 0, \forall x \in C$;

$(A2)$ ϕ is monotone, i.e.,

$$\phi(x, y) + \phi(y, x) \leq 0, \forall x, y \in C;$$

$(A3)$ for any $x, y, z \in C$ the functional $x \longmapsto \phi(x, y)$ is upper hemicontinuous, i.e.,

$$\lim_{t \to 0^+} \sup \phi(tz + (1-t)x, y) \leq \phi(x, y),\ \forall x, y, z \in C;$$

(A4) $y \longmapsto \phi(x, y)$ is convex and weakly lower semi-continuous.

The following lemmas will be needed in proving our main results :

Lemma 2.1 ([1]). Let H be a real Hilbert space, C be nonempty closed convex subset of H, $\phi : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying the condition (A1) - (A4), then , for any given $x \in H$ and $r > 0$, there exists $z \in C$ such that

$$\phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C.$$

Lemma 2.2 ([6]). Suppose all conditions in Lemma 2.1 are satisfied. For any given $r > 0$ define a mapping $T_r : H \longrightarrow C$ as follows:

$$T_r(x) = \{z \in C : \phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}, x \in H. \qquad (2.2.5)$$

Then the following conclusions hold :

(1) T_r is single-valued :

(2) T_r is firmly nonexpansive, i.e.,

$$||T_r x - T_r y||^2 \leq \langle T_r x - T_r y, x - y\rangle, \forall x, y \in H.$$

This implies that $||T_r x - T_r y|| \leq ||x - y||, \forall x, y \in H$, i.e.,

T_r is a nonexpansive mapping.

(3) $F(T_r)$ = $EP(\phi)$, $\forall r > 0$;

(4) $EP(\phi)$ is a closed and convex set.

Lemma 2.3 ([12]). Let X be a Banach space ,$\{x_n\}$, $\{y_n\}$ be two bounded sequence in X and $\{\beta_n\}$ be a sequence in $[0, 1]$ satisfying

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n, \forall n \geq 1$ and

$$\limsup_{n\to\infty}\{||y_{n+1} - y_n|| - ||x_{n+1} - x_n||\} \leq 0,$$

then $\lim\limits_{n\to\infty} ||y_n - x_n|| = 0.$

Lemma 2.4 ([16]). Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative real sequences satisfying the following condition :

$$a_{n+1} \leq (1 - \gamma_n)a_n + b_n, \forall n \geq n_0,$$

where n_0 is some nonnegative integer , $\{\gamma_n\}$ is a sequence in (0,1) and $\{b_n\}$ is a sequence in $\mathbb{R}$ such that :

(i) $\sum\limits_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup\limits_{n\to\infty} \dfrac{b_n}{\gamma_n} \leq 0$ or $\sum\limits_{n=1}^{\infty} |b_n| < \infty.$
Then $\lim\limits_{n\to\infty} a_n = 0$.

Definition 2.1 ([10]). Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings and $\{\mu_i\}$ be a nonnegative real sequence with 0$\leq$ $\mu_i < 1, \forall i \geq 1$. For any $n \geq 1$define a mapping $W_n : C \to C$ as follows:

$$\begin{cases} U_{n,n+1} = 1, \\ U_{n,n} = \mu_n S_n U_{n,n+1} + (1-\mu_n) I, \\ U_{n,n-1} = \mu_{n-1} S_{n-1} U_{n,n} + (1-\mu_{n-1}) I, \\ . \\ . \\ U_{n,k} = \mu_k S_k U_{k+1} + (1-\mu_k) I, \\ U_{n,k-1} = \mu_{k-1} S_{k-1} U_{n,k} + (1-\mu_{k-1}) I, \\ . \\ . \\ U_{n,2} = \mu_2 S_2 U_{n,3} + (1-\mu_2) I, \\ W_n = U_{n,1} = \mu_1 S_1 U_{n,2} + (1-\mu_1) I. \end{cases} \tag{2.2.6}$$

Such a mapping W_n is nonexpansive from C to C and it is called a W-mapping generated by $S_n, S_{n-1}, ..., S_1$ and $\mu_n, \mu_{n-1}, ..., \mu_1$.

Lemma 2.5 (Shinoji et at.[10]). Let C be a nonempty closed convex subset of a Hilbert space H , $S_i : C \to C$ be a family of infinitely nonexpansive mappings with $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset, \{\mu_i\}$ be a real sequence such that $0 < \mu_i \leq b < 1$, $\forall i \geq 1$. Then,
(1) W_n is nonexpansive and $F(W_n) = \bigcap_{i=1}^{n} F(S_i)$ for each $n \geq 1$;
(2) for each $x \in C$ and for each positive integer k , the limit $\lim_{n\to\infty} U_{n,k}x$ exists;
(3) the mapping $W : C \to C$ defined by

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, x \in C, \tag{2.2.7}$$

is a nonexpansive mapping satisfying $F(W) = \bigcap_{i=1}^{\infty} F(S_i)$ and it is called the W-mapping generated by $S_1, S_2, ...$ and $\mu_1, \mu_2, ...$

Lemma 2.6 Let C be a nonempty closed convex subset of a Hilbert space H , $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings with $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset, \{\mu_i\}$ be a real sequence such that$0 < \mu_i \leq b < 1, \forall i \geq 1$. If K is any bounded subset of C , then

$$\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0. \tag{2.2.8}$$

Proof. Take $p \in \bigcap_{i=1}^{\infty} F(S_i)$. Since K is a bounded subset of C , there exists an $M > 0$ such that $\sup_{x \in K} \|x - p\| \leq M$. Hence for any $n \geq 1$ and $x \in K$, we have

$$\begin{aligned}
\|W_{n+1}x - W_n x\| &= \|U_{n+1,1}x - U_{n,1}x\| \\
&= \|\mu_1 S_1 U_{n+1,2}x + (1-\mu_1)x - (\mu_1 S_1 U_{n,2}x + (1-\mu_1)x)\| \\
&\leq \mu_1 \|U_{n+1,2}x - U_{n,2}x\| \\
&= \mu_1 \|\mu_2 S_2 U_{n+1,3}x + (1-\mu_2)x - \mu_2 S_2 U_{n,3}x - (1-\mu_2)x\| \\
&\leq \mu_1 \mu_2 \|U_{n+1,3}x - U_{n,3}x\| \\
&. \\
&. \\
&. \\
&\leq (\prod_{i=1}^{n} \mu_i) \|U_{n+1,n+1}x - U_{n,n+1}x\| \\
&= (\prod_{i=1}^{n} \mu_i) \|\mu_{n+1} S_{n+1} U_{n+1,n+2}x + (1-\mu_{n+1})x - x\| \\
&= (\prod_{i=1}^{n+1} \mu_i)(\|S_{n+1}x - x\|) \\
&\leq \prod_{i=1}^{n+1} (\|S_{n+1}x - p\| + \|p - x\|) \\
&\leq 2 \prod_{i=1}^{n+1} \mu_i \|x - p\| \leq 2(\prod_{i=1}^{n+1} \mu_i) M. \qquad (2.2.9)
\end{aligned}$$

Since $0 < \mu_i \leq b < 1$, for any given $\epsilon > 0$, there exists a positive integer n_o such that

$$b^{n_0+1} < \frac{\epsilon(1-b)}{2M}.$$

Hence for any positive integers $m > n > n_0$, from (2.2.9) we have

$$\|W_m x - W_n x\| \le \sum_{j=n}^{m-1} \|W_{j+1} W_j x\| \le \sum_{j=n}^{m-1} 2M(\prod_{j=1}^{j+1} \mu_i)$$
$$\le 2M \sum_{j=n}^{m-1} b^{j+1} \le \frac{2Mb^{n+1}}{1-b} < \epsilon, \forall x \in K. \tag{2.2.10}$$

In (2.2.10) letting $m \to \infty$, for any $x \in K$, we have

$$\|Wx - W_n x\| \le \epsilon, \forall n > n_0.$$

Therefore, we have

$$\sup_{x \in K} \|Wx - W_n x\| \le \epsilon, \forall n > n_0.$$

This implies that $\lim_{n\to\infty} \sup_{x \in K} \|Wx - W_n x\| = 0$.

The conclusion of Lemma 2.6 is proved.

CHAPTER III

Main results

3.1 Main Result

In this section, we shall extend the proof line for the strong convergence theorem proposed by S. Chang et al. The iterative algorithm shown in this Theorem is constructed by using the viscosity approximation method for finding a common element of the set of common fixed points for a family of infinitely nonexpansive mapping , the set of solutions of the variational inequality for an α-inverse-strongly monotone mapping and the set of solutions of an equilibrium problem in Hilbert space.

Theorem 3.1.1 ([17]) Let *H* be real Hilbert space , *C* be a nonempty closed convex subset of H , $\phi : C \times C \to \mathbb{R}$ be a bifunction satisfying the condition (A1) - (A4) , $A : C \to H$ an α - *inverse-strongly monotone mapping* ,$\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings with $F \cap VI(C, A) \cap EP(\phi) \neq \emptyset$, where $F := \bigcap_{i=1}^{\infty} F(S_i)$ and $f : H \to H$ be a contraction mapping with a contractive constant $\xi \in (0, 1)$.

Let $\{x_n\}, \{y_n\}, \{k_n\}$ and $\{u_n\}$ be the sequence defined by

$$\begin{cases} \phi(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n k_n, \\ k_n = P_C(y_n - \lambda_n A y_n), \\ y_n = P_C(u_n - \lambda_n A u_n), \end{cases} \tag{3.1.1}$$

where $W_n : C \to C$ is the sequence defined by (2.2.6) ,$\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in [0 , 1] , λ_n is a sequence in $[a, b] \subset (0, 2\alpha)$ and $\{r_n\}$ is a sequence in $(0, \infty)$. If the following conditions are satisfied :

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0 \quad , \sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iv) $\liminf_{n\to\infty} r_n > 0, \qquad \sum_{n=1}^{\infty} | r_{n+1} - r_n | < \infty$;

(v) $\limsup_{n\to\infty} | \lambda_{n+1} - \lambda_n | = 0$;

then $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F \cap VI(C, A) \cap EP(\phi)$, where $z = P_{F\cap VI(C,A)\cap EP(\phi)} f(z)$.

Proof. We divide the proof of Theorem 3.1 into six steps:

(I) First , we prove that there exists $z \in C$ such that $z = P_{F\cap VI(C,A)\cap EP(\phi)} f(z)$.

In fact, since $f : H \to H$ is a contraction with a contractive constant ξ, hence $P_{F\cap VI(C,A)\cap EP(\phi)} f : H \to C$ is also a contraction, and

$$\|P_{F\cap VI(C,A)\cap EP(\phi)} f(x) - P_{F\cap VI(C,A)\cap EP(\phi)} f(y)\| \leq \xi\|x - y\|, \;\; \forall x, y \in H.$$

By the Banach theorem, there exists a unique $z \in C$ such that $z = P_{F\cap VI(C,A)\cap EP(\phi)} f(z)$.

(II) Now we prove that the sequences $\{x_n\}, \{u_n\}$ and $\{k_n\}$ are bounded.

In fact, since $A : C \to H$ is α-inverse-strongly monotone, for any $x, y \in C$ and $\lambda_n \in [a, b] \subset [0, 2\alpha]$, we have

$$\begin{aligned}
\|(I - \lambda_n)x - (I - \lambda_n)y\|^2 &= \|(x - y) - \lambda_n(Ax - Ay)\|^2 \\
&= \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay\rangle + \lambda_n^2\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 - 2\lambda_n\alpha\|Ax - Ay\|^2 + \lambda_n^2\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2, \qquad (3.1.2)
\end{aligned}$$

which implies that $I - \lambda_n A$ is nonexpansive. Let $x^* \in F \cap VI(C, A) \cap EP(\phi)$ and let $\{T_{r_n}\}$be the sequence of mappings defined by (2.2.5) . It follows from Lemma2.2 and (2.2.3) that $x^* = P_c(x^* - \lambda_n Ax^*) = T_{r_n}x^*$ and $u_n = T_{r_n}x_n$. From (3.1.2) , we have

$$\begin{aligned}\|k_n - x^*\| &= \|P_c(y_n - \lambda_n A y_n) - P_c(x^* - \lambda_n A x^*)\| \\ &\leq \|(y_n - \lambda_n A y_n) - (x^* - \lambda_n A x^*)\| \leq \|y_n - x^*\| \\ &= \|P_c(u_n - \lambda_n A u_n) - P_c(x^* - \lambda_n A x^*)\| \\ &\leq \|(u_n - \lambda_n A u_n) - (x^* - \lambda_n A x^*)\| \leq \|u_n - x^*\| \\ &= \|T_{r_n} x_n - T_{r_n} x^*\| \leq \|x_n - x^*\|.\end{aligned}$$

This implies that

$$\|k_n - x^*\| \leq \|y_n - x^*\| \leq \|u_n - x^*\| \leq \|x_n - x^*\|. \tag{3.1.3}$$

By Lemma 2.5, $x^* = W_n x^*$, hence from (3.1.3) we have

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n k_n - W_n x^*\| \\ &\leq \alpha_n \|f(x_n) - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|k_n - x^*\| \\ &\leq \alpha_n \|f(x_n) - f(x^*) + f(x^*) - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|x_n - x^*\| \\ &\leq \alpha_n (\|f(x_n) - f(x^*)\| + \|f(x^*) - x^*\|) + (\beta_n + \gamma_n)\|x_n - x^*\| \\ &= \alpha_n \|f(x_n) - f(x^*)\| + \alpha_n \|f(x^*) - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \\ &\leq \alpha_n \xi \|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \\ &= (\alpha_n \xi + (1 - \alpha_n))\|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| \\ &= (1 - (\alpha_n - \alpha_n \xi)\|x_n - x^*\| + (1 - \xi)\alpha_n \frac{1}{1 - \xi}\|f(x^*) - x^*\| \\ &= (1 - (1 - \xi)\alpha_n)\|x_n - x^*\| + (1 - \xi)\alpha_n \frac{1}{1 - \xi}\|f(x^*) - x^*\| \\ &\leq \max\{\|x_n - x^*\|, \frac{1}{1 - \xi}\|f(x^*) - x^*\| \\ &\quad . \\ &\quad . \\ &\quad . \\ &\leq \max\{\|x_1 - x^*\|, \frac{1}{1 - \xi}\|f(x^*) - x^*\| \quad \forall n \geq 1.\end{aligned}$$

This shows that $\{x_n\}$ is bounded. From (3.1.3) we know that $\{u_n\}, \{k_n\}, \{y_n\}, \{W_n k_n\}$, $\{A(u_n)\}, \{A y_n\}$ and $\{f(x_n)\}$ all are bounded. Especially, $\{u_n\}, \{k_n\}, \{y_n\}, \{W_n k_n\}$ are bounded sequences in C. Without loss of generality, we can assume that there exists a bounded set $K \subset C$ such that

$$u_n, k_n, y_n \in K, \quad \forall n \geq 1. \tag{3.1.4}$$

(III) Next we prove that $\|x_{n+1} - x_n\| \to 0$.

In fact, since $I - \lambda_n A$ is nonexpansive, we have

$$\begin{aligned}
\|k_{n+1} - k_n\| &= \|P_C(y_{n+1} - \lambda_{n+1}Ay_{n+1}) - P_C(y_n - \lambda_n Ay_n)\| \\
&\leq \|(y_{n+1} - \lambda_{n+1}Ay_{n+1}) - (y_n - \lambda_n Ay_n)\| \\
&= \|(y_{n+1} - \lambda_{n+1}Ay_{n+1}) - (y_n - \lambda_{n+1}Ay_n + \lambda_{n+1}Ay_n - \lambda_n Ay_n)\| \\
&= \|(y_{n+1} - \lambda_{n+1}Ay_{n+1}) - (y_n - \lambda_{n+1}Ay_n) + (\lambda_n Ay_n - \lambda_{n+1}Ay_n)\| \\
&\leq \|(y_{n+1} - \lambda_{n+1}Ay_{n+1}) - (y_n - \lambda_{n+1}Ay_n)\| + \|\lambda_n Ay_n - \lambda_{n+1}Ay_n\| \\
&\leq \|(y_{n+1} - \lambda_{n+1}Ay_{n+1}) - (y_n - \lambda_{n+1}Ay_n)\| + |\lambda_n - \lambda_{n+1}|\|Ay_n\| \\
&\leq \|y_{n+1} - y_n\| + |\lambda_n - \lambda_{n+1}|\|Ay_n\| \\
&= \|P_C(u_{n+1} - \lambda_{n+1}Au_{n+1}) - P_C(u_n - \lambda_n Au_n)\| + |\lambda_n - \lambda_{n+1}|\|Ay_n\| \\
&\leq \|(u_{n+1} - \lambda_{n+1}Au_{n+1}) - (u_n - \lambda_n Au_n)\| + |\lambda_n - \lambda_{n+1}|\|Ay_n\| \\
&\leq \|(u_{n+1} - \lambda_{n+1}Au_{n+1}) - (u_n - \lambda_{n+1}Au_n)\| + |\lambda_n - \lambda_{n+1}|(\|Au_n\| + \\
&\quad \|Ay_n\|) \\
&\leq \|(u_{n+1} - u_n\| + |\lambda_n - \lambda_{n+1}|(\|Au_n\| + \|Ay_n\|).
\end{aligned} \tag{3.1.5}$$

By Lemma 2.2 $u_n = T_{r_n}x_n, u_{n+1} = T_{r_{n+1}}x_{n+1}$, we have

$$\phi(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0, \ \ \forall y \in C, \tag{3.1.6}$$

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \ \forall y \in C. \tag{3.1.7}$$

Take $y = u_{n+1}$ in(3.1.7) and $y = u_n$ in (3.1.6), then add these two inequalities. By using condition (A2), we have

$$\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \rangle \geq 0.$$

Hence

$$\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \rangle \geq 0.$$

This implies that

$$\begin{aligned}\|u_{n+1} - u_n\|^2 &\leq \langle u_{n+1} - u_n, x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1}) \rangle \\ &\leq \|u_{n+1} - u_n\| \{ \|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}| \cdot \|u_{n+1} - x_{n+1}\| \}.\end{aligned}$$

From condition (iv), without loss of generality, we can assume that $r_n > c$, $\forall n \geq 1$, hence we have

$$\begin{aligned}\|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}| \|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{c}|r_{n+1} - r_n| \cdot M, \qquad (3.1.8)\end{aligned}$$

where $M = \sup_{n \geq 1}\{\|u_n - x_n\|\}$. Substituting (3.1.8) into (3.1.5), we have

$$\begin{aligned}\|k_{n+1} - k_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{c}|r_{n+1} - r_n| \cdot M \\ &+ |\lambda_n - \lambda_{n+1}|(\|Au_n\| + \|Ay_n\|). \qquad (3.1.9)\end{aligned}$$

Letting $x_{n+1} = (1 - \beta)z_n + \beta_n x_n, \forall n \geq 1$, then we have

$$\begin{aligned}z_{n+1} - z_n &= \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} W_{n+1} k_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n W_n k_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(f(x_{n+1}) - f(x_n)) + (\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n})f(x_n) \\ &+ \frac{\gamma_{n+1}}{1 - \beta_{n+1}}(W_{n+1}k_{n+1} - W_n k_n) + (\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n})W_n k_n.\end{aligned}$$

By condition (i),

$$\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n} = \frac{\alpha_n}{1-\beta_n} - \frac{\alpha_{n+1}}{1-\beta_{n+1}},$$

and so we have

$$\begin{aligned} \|z_{n+1}-z_n\| &\le \frac{\xi\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|\{\|f(x_n)\| + \|W_nk_n\|\} \\ &+ \frac{\gamma_{n+1}}{1-\beta_{n+1}}\|W_{n+1}k_{n+1}-W_nk_n\|. \end{aligned} \tag{3.1.10}$$

It follows from (3.1.9) that

$$\begin{aligned} \|W_{n+1}k_{n+1}-W_nk_n\| &= \|W_{n+1}k_{n+1}-Wk_{n+1}+Wk_{n+1}-Wk_n+Wk_n-W_nk_n\| \\ &\le \|W_{n+1}k_{n+1}-Wk_{n+1}\| + \|Wk_{n+1}-Wk_n\| + \|Wk_n-W_nk_n\| \\ &\le \sup_{x\in K}\{\|Wx-W_{n+1}x\| + \|Wx-W_nx\|\} + \|k_{n+1}-k_n\| \\ &\le \sup_{x\in K}\{\|Wx-W_{n+1}x\| + \|Wx-W_nx\|\} + \|x_{n+1}-x_n\| \\ &+ \frac{1}{c}|r_{n+1}-r_n|\cdot M + |\lambda_n-\lambda_{n+1}|(\|Au_n\|+\|Ay_n\|). \end{aligned} \tag{3.1.11}$$

where K is the bounded subset of C defined by (3.1.4). By Lemma 2.6 we know that

$$\sup_{x\in K}\|Wx-W_nx\| \to 0 \ \ (as\ \ n\to\infty). \tag{3.1.12}$$

Substituting (3.1.11) into (3.1.10), after simplifying we have

$$\begin{aligned} \|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| \le & \left\{\frac{\xi\alpha_{n+1}+\gamma_{n+1}}{1-\beta_{n+1}} - 1\right\}\|x_{n+1}-x_n\| + \\ & \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| \times \{\|f(x_n)\| + \|W_nk_n\|\} + \\ & \frac{\gamma_{n+1}}{1-\beta_{n+1}}\sup_{x\in K}\|Wx-W_{n+1}x\| + \|Wx-W_nx\|\} + \\ & \frac{1}{c}|r_{n+1}-r_n|\cdot M + |\lambda_n-\lambda_{n+1}|(\|Au_n\|+\|Ay_n\|). \end{aligned}$$

since $\dfrac{\xi\alpha_{n+1}+\gamma_{n+1}}{1-\beta_{n+1}}-1=\dfrac{(\xi-1)\alpha_{n+1}}{1-\beta_{n+1}}<0$, by conditions (ii)-(v), we have

$$\limsup_{n\to\infty}\{\|Z_{n+1}-Z_n\|-\|x_{n+1}-x_n\|\}\le 0.$$

Hence from Lemma 2.3 we have

$$\lim_{n\to\infty}\|z_n-x_n\|=0.$$

Consequently,

$$\begin{aligned}\lim_{n\to\infty}\|x_{n+1}-x_n\| &= \lim_{n\to\infty}\|\beta_n x_n+(1-\beta_n)z_n-x_n\|\\ &= \lim_{n\to\infty}\|(-x_n)(1-\beta_n)+(1-\beta_n)z_n\|\\ &= \lim_{n\to\infty}(1-\beta_n)\|-x_n+z_n\|\\ &= \lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\|=0. \qquad (3.1.13)\end{aligned}$$

(IV) Now we prove that $\|W_nk_n-k_n\|\to 0$. Since

$$\|W_nk_n-k_n\|\le\|W_nk_n-x_n\|+\|x_n-u_n\|+\|u_n-k_n\|,$$

for the above purpose, it is sufficient to prove $\|W_nk_n-x_n\|\to 0$, $\|x_n-u_n\|\to 0, \|u_n-k_n\|\to 0$.

(a) First we prove that $\|x_n-W_nk_n\|\to 0$. In fact, since $\alpha_n\to 0$ and $\|x_{n+1}-x_n\|\to 0$ we have

$$\begin{aligned}\|x_n-W_nk_n\| &\le \|x_n-x_{n+1}\|+\|x_{n+1}-W_nk_n\|\\ &= \|x_n-x_{n+1}\|+\|\alpha_nf(x_n)+\beta_nx_n+\gamma_nW_nk_n-W_nk_n\|\\ &= \|x_n-x_{n+1}\|+\|\alpha_nf(x_n)+\beta_nx_n+(1-\alpha_n-\beta_n)W_nk_n-W_nk_n\|\\ &= \|x_n-x_{n+1}\|+\|\alpha_nf(x_n)+\beta_nx_n+W_nk_n-\alpha_nW_nk_n-\\ &\quad\ \beta_nW_nk_n-W_nk_n\|\\ &= \|x_n-x_{n+1}\|+\|\alpha_nf(x_n)-\alpha_nW_nk_n)+(\beta_nx_n-\beta_nW_nk_n)\|\\ &\le \|x_n-x_{n+1}\|+\alpha_n\|\|f(x_n)-W_nk_n\|+\beta_n\|x_n-W_nk_n\|.\end{aligned}$$

Simplifying it, we have

$$\|x_n - W_n k_n\| \leq \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|f(x_n) - W_n k_n\| \to 0 \quad (3.1.14)$$

as $(n \to \infty)$.

(b) Next we prove that $\|x_n - u_n\| \to 0$,

$$\begin{aligned} \|u_n - v\|^2 = \|T_{r_n}x_n - T_{r_n}v\|^2 &\leq \langle T_{r_n}x_n - T_{r_n}v, x_n - v\rangle = \langle u_n - v, x_n - v\rangle \\ &= \frac{1}{2}(\|u_n - v\|^2 + \|x_n - v\|^2 - \|u_n - x_n\|^2), \end{aligned}$$

and hence

$$\|u_n - v\|^2 \leq \|x_n - v\|^2 - \|u_n - x_n\|^2. \quad (3.1.15)$$

By virtue of the convexity of norm $\|.\|^2$ and (3.1.3), we have

$$\begin{aligned} \|x_{n+1} - v\|^2 &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n k_n - v\|^2 \\ &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n k_n - (\alpha_n + \beta_n + \gamma_n)v\|^2 \\ &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n x_n - \alpha_n v - \beta_n v + \gamma_n v\|^2 \\ &\leq \|\alpha_n f(x_n) - \alpha_n v\|^2 + \|\beta_n x_n - \beta_n v\|^2 + \|\gamma_n W_n k_n - \gamma_n v\|^2 \\ &\leq \alpha_n\|f(x_n) - v\|^2 + \beta_n\|x_n - v\|^2 + \gamma_n\|W_n k_n - v\|^2 \\ &\leq \alpha_n\|f(x_n) - v\|^2 + \beta_n\|x_n - v\|^2 + \gamma_n\|k_n - v\|^2 \\ &\leq \alpha_n\|f(x_n) - v\|^2 + \beta_n\|x_n - v\|^2 + \gamma_n\|y_n - v\|^2. \quad (3.1.16) \end{aligned}$$

Again by (3.1.3) and (3.1.15)

$$\|y_n - v\|^2 \leq \|u_n - v\|^2 \leq \|x_n - v\|^2 - \|u_n - x_n\|^2.$$

Substituting the above inequality into (3.1.16), we have

$$\begin{aligned}\|x_{n+1}-v\|^2 &\le \alpha_n\|f(x_n)-v\|^2+\beta_n\|x_n-v\|^2+\gamma_n\{\|x_n-v\|^2-\|u_n-x_n\|^2\}\\ &= \alpha_n\|f(x_n)-v\|^2+\beta_n\|x_n-v\|^2+\gamma_n\|x_n-v\|^2-\gamma_n\|u_n-x_n\|^2\\ &= \alpha_n\|f(x_n)-v\|^2+(\beta_n+\gamma_n)\|x_n-v\|^2-\gamma_n\|u_n-x_n\|^2\\ &\le \alpha_n\|f(x_n)-v\|^2+(1-\alpha_n)\|x_n-v\|^2-\gamma_n\|u_n-x_n\|^2,\end{aligned}$$

and hence

$$\begin{aligned}\gamma_n\|u_n-x_n\|^2 &\le \alpha_n\|f(x_n)-v\|^2+\|x_n-v\|^2-\|x_{n+1}-v\|^2\\ &\le \alpha_n\|f(x_n)-v\|^2+(\|x_n-x_{n+1}\|)(\|x_n-v\|+\|x_{n+1}-v\|).\end{aligned}$$

By virtue of conditions (i)-(iii), $\alpha_n \to 0$ and

$$\liminf_{n\to\infty}\gamma_n=\liminf_{n\to\infty}(1-\alpha_n-\beta_n)=1-\limsup_{n\to\infty}\beta_n>0.$$

Therefore from (3.1.13)

$$\|x_n-u_n\|\to 0 \quad (n\to\infty). \tag{3.1.17}$$

(c) Next we prove that $\|u_n-k_n\|\to 0$.

In fact, since A is α-inverse-strongly monotone, by the assumptions imposed on $\{\lambda_n\}$ and (3.1.3) for given $v\in F\cap VI(C,A)\cap EP(\phi)$ we have

$$\begin{aligned}\|y_n-v\|^2 &= \|(I-\lambda_nA)u_n-(I-\lambda_nA)v\|^2\\ &= \|u_n-\lambda_nAu_n-v+\lambda_nAv\|^2\\ &= \|(u_n-v)-\lambda_n(Au_n-Av)\|^2\\ &= \|u_n-v\|^2-2\lambda_n\langle u_n-v,Au_n-Av\rangle+\lambda_n^2\|Au_n-Av\|^2\\ &\le \|u_n-v\|^2-2\lambda_n\alpha\|Au_n-Av\|+\lambda_n^2\|Au_n-Av\|^2\\ &= \|u_n-v\|^2+\lambda_n(\lambda_n-2\alpha)\|Au_n-Av\|^2\\ &\le \|x_n-v\|^2+a(b-2\alpha)\|Au_n-Av\|^2.\end{aligned}$$

Substituting it into (3.1.16) , we have

$$\begin{aligned}\|x_{n+1}-v\|^2 &\le \alpha_n\|f(x_n)-v\|^2+\beta_n\|x_n-v\|^2+\gamma_n\{\|x_n-v\|^2+\\ &\qquad a(b-2\alpha)\|Au_n-Av\|^2\}\\ &= \alpha_n\|f(x_n)-v\|^2+(\beta_n+\gamma_n)\|x_n-v\|^2+a(b-2\alpha)\gamma_n\|Au_n-Av\|^2\end{aligned}$$

$$= \alpha_n\|f(x_n) - v\|^2 + (1-\alpha_n)\|x_n - v\|^2 + a(b-2\alpha)\gamma_n\|Au_n - Av\|^2,$$

i.e.,

$$a(2\alpha - b)\gamma_n\|Au_n - Av\|^2 \leq \alpha_n\|f(x_n) - v\|^2 + \|x_n - v\|^2 - \|x_{n+1} - v\|^2$$
$$\leq \alpha_n\|f(x_n) - v\|^2 + (\|x_n - v\| + \|x_{n+1} - v\|)\|x_{n+1} - x_n\|.$$

Since $\alpha_n \to 0, a, b \in (0, 2\alpha), \liminf_{n\to\infty} \gamma_n > 0$ and $\|x_{n+1} - x_n\| \to 0$, we obtain

$$\|Au_n - Av\| \to 0 (n \to \infty). \tag{3.1.18}$$

Again from (2.2.1) and (3.1.3) , for any $v \in F \cap EP(\phi) \cap VI(C, A))$, we have

$$\begin{aligned}
\|y_n - v\|^2 &= \|P_C(u_n - \lambda_n Au_n) - P_C(v - \lambda_n Av)\|^2 \\
&\leq \langle (u_n - \lambda_n Au_n) - (v - \lambda_n Av), y_n - v\rangle \\
&= \frac{1}{2}\{\|u_n - \lambda_n Au_n) - (v - \lambda_n Av)\|^2 + \|y_n - v\|^2 - \|(u_n - \lambda_n Au_n) - \\
&\quad (v - \lambda_n Av) - (y_n - v)\|^2\} \\
&= \frac{1}{2}\{\|(I - \lambda_n A)u_n - (I - \lambda_n A)v\|^2 + \|y_n - v\|^2 - \|u_n - \lambda_n Au_n \\
&\quad - v + \lambda_n Av - y_n + v\|^2 \\
&\leq \frac{1}{2}\{\|u_n - v\|^2 + \|y_n - v\|^2 - \|u_n - y_n - \lambda_n(Au_n - Av)\|^2\} \\
&\leq \frac{1}{2}\{\|u_n - v\|^2 + \|y_n - v\|^2 - \|u_n - y_n\|^2 + 2\lambda_n\langle u_n - y_n, Au_n - \\
&\quad Av\rangle - \lambda_n^2\|Au_n - Av\|^2\}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|y_n - v\|^2 &\leq \|u_n - v\|^2 - \|u_n - y_n\|^2 + 2\lambda_n\langle u_n - y_n, Au_n - Av\rangle - \lambda_n^2\|Au_n - Av\|^2 \\
&\leq \|x_n - v\|^2 - \|u_n - y_n\|^2 + 2\lambda_n\langle u_n - y_n, Au_n - Av\rangle \\
&- \lambda_n^2\|Au_n - Av\|^2.
\end{aligned} \tag{3.1.19}$$

Substituting (3.1.19)into (3.1.16), we have

$$\|x_{n+1} - v\|^2 \leq \alpha_n\|f(x_n) - v\|^2 + \beta_n\|x_n - v\| + \gamma_n\{\|x_n - v\|^2 - \|u_n - y_n\|^2 +$$
$$2\lambda_n\langle u_n - y_n, Au_n - Av\rangle - \lambda_n^2\|Au_n - Av\|^2$$

$$\leq \alpha_n\|f(x_n)-v\|^2+\|x_n-v\|-\gamma_n\|u_n-y_n\|^2+2\gamma_n\lambda_n\langle u_n-y_n, Au_n-Av\rangle-\gamma_n\lambda_n^2\|Au_n-Av\|^2,$$

which implies that

$$\gamma_n\|u_n-y_n\|^2 \leq \alpha_n\|f(x_n)-v\|^2+\|x_n-v\|^2-\|x_{n+1}-v\|^2+2\gamma_n\lambda_n\|u_n-y_n\|\|Au_n-Av\|.$$

By conditions (i)-(iii), $\alpha_n \to 0$ and $\liminf_{n\to\infty}\gamma_n > 0$. Hence it follows from $\|x_{n+1}-x_n\| \to 0$ and $\|Au_n-Av\| \to 0$ that $\|u_n-y_n\| \to 0 (n \to \infty)$.

Since $\|k_n y_n\| = \|P_C(y_n-\lambda_n Ay_n)-P_C(u_n-\lambda_n Au_n)\| \leq \|y_n-u_n\|$,we have

$$\|k_n-u_n\| \leq \|k_n-y_n\|+\|y_n-u_n\| \leq 2\|y_n-u_n\| \to 0 (n\to\infty), \tag{3.1.20}$$

which together with (3.1.14) and(3.1.17) shows that

$$\|W_nk_n-k_n\| \leq \|W_nk_n-x_n\|+\|x_n-u_n\|+\|u_n-k_n\| \to 0 (n\to\infty). \tag{3.1.21}$$

(V) Next, we prove that $\limsup_{n\to\infty}\langle f(z)-z, x_n-z\rangle \leq 0$,

where $z = P_{F\cap VI(C,A)\cap EP(\phi)}f(z)$. For this purpose, we choose a subsequence $\{k_{n_i}\} \subset \{k_n\}$ such that

$$\limsup_{n\to\infty}\langle f(z)-z, W_nk_n-z\rangle = \lim_{i\to\infty}\langle f(z)-z, W_{n_i}k_{n_i}-z\rangle.$$

Since$\{k_{n_i}\}$ is bounded in C, without loss of generality, we can assume that $k_{n_i} \rightharpoonup k \in C$. Since $\|W_{n_i}k_{n_i}-k_{n_i}\| \to 0$, this implies that $W_{n_i}k_{n_i} \rightharpoonup k$. Therefore we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f(z)-z, W_nk_n-z\rangle &= \lim_{i\to\infty}\langle f(z)-z, W_{n_i}k_{n_i}-z\rangle \\ &= \langle f(z)-z, k-z\rangle. \end{aligned}\tag{3.1.22}$$

Next we prove that $k \in F\cap VI(C,A)\cap EP(\phi)$.

(a) First, we prove $k \in VI(C, A)$.

$$Tx = \begin{cases} Ax + N_c x, & x \in C; \\ \emptyset, \quad x \notin C. \end{cases}$$

For any given$(x, u) \in G(T)$. hence $u - Ax \in N_C x$. Since $k_n \in C$, by the definition of N_C, we have

$$\langle x - k_n, u - Ax \rangle \geq 0. \tag{3.1.23}$$

On the other hand, since $k_n = P_C(y_n - \lambda_n A y_n)$, we have $\langle x - k_n, k_n - (y_n - \lambda_n A y_n) \rangle \geq 0$, and so

$$\langle x - k_n, \frac{k_n - y_n}{\lambda_n} + Ay_n \rangle \geq 0.$$

By (3.1.23) and the α-inverse monotonicity of A, we have

$$\begin{aligned} \langle x - k_{n_i}, u \rangle &\geq \langle x - k_{n_i}, Ax \rangle \\ &\geq \langle x - k_{n_i}, Ax \rangle - \langle x - k_{n_i}, \frac{k_{n_i} - y_{n_i}}{\lambda_{n_i}} + Ay_{n_i} \rangle \\ &= \langle x - k_{n_i}, Ax - Ak_{n_i} \rangle + \langle x - k_{n_i}, Ak_{n_i} - Ay_{n_i} \rangle - \langle x - k_{n_i}, \frac{k_{n_i} - y_{n_i}}{\lambda_{n_i}} \rangle \\ &\geq \langle x - k_{n_i}, Ak_{n_i} - Ay_{n_i} \rangle - \langle x - k_{n_i}, \frac{k_{n_i} - y_{n_i}}{\lambda_{n_i}} \rangle. \end{aligned}$$

Since $\|k_n - y_n\| \to 0, k_{n_i} \rightharpoonup k$ and A is Lipschitz continuous, we have

$$\lim_{i \to \infty} \langle x - k_{n_i}, u \rangle = \langle x - k, u \rangle \geq 0.$$

Again, since T is maximal monotone, hence $0 \in Tk$. This shows that $k \in VI(C, A)$ (see[7]).

(b) Next we prove that $k \in F(W) = \bigcap_{n=1}^{\infty} F(S_n)$

Suppose the contrary, $k \notin F(W)$, i.e., $Wk \neq k$. Since $k_{n_i} \rightharpoonup k$, by the Opial condition, we have

$$\begin{aligned}\liminf_{i\to\infty}\|k_{n_i}-k\| &< \liminf_{i\to\infty}\|k_{n_i}-Wk\| \\ &\leq \liminf_{i\to\infty}\{\|k_{n_i}-Wk_{n_i}\|+\|Wk_{n_i}-Wk\|\} \\ &= \liminf_{i\to\infty}\{\|k_{n_i}-Wk_{n_i}\|+\|k_{n_i}-k\|\}. \qquad (3.1.24)\end{aligned}$$

By (2.2.8) and (3.1.21)

$$\begin{aligned}\lim_{i\to\infty}\|Wk_{n_i}-k_{n_i}\| &\leq \lim_{i\to\infty}\{\|Wk_{n_i}-W_{n_i}k_{n_i}\|+\|W_{n_i}k_{n_i}-k_{n_i}\|\} \\ &\leq \lim_{i\to\infty}\{\sup_{x\in K}\|Wx-W_{n_i}x\|\} \\ &+ \lim_{i\to\infty}\|W_{n_i}k_{n_i}-k_{n_i}\|=0, \qquad (3.1.25)\end{aligned}$$

therefore, we have

$$\liminf_{i\to\infty}\|k_{n_i}-k\| < \lim_{i\to\infty} inf\|k_{n_i}-k\|$$

This is a contradiction, which shows that $k\in F(W)=F=\bigcap_{n=1}^{\infty}F(S_n)$.

(c) Now we prove that $k\in EP(\phi)$.

$$\phi(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\forall y\in C.$$

By condition (A2)

$$\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq\phi(y,u_n),$$

and hence

$$\langle y-u_n,\frac{u_n-x_{n_i}}{r_{n_i}}\rangle\geq\phi(y,u_{n_i}) \qquad (3.1.26)$$

on the other hand, from (3.1.20) we have

$$|||u_n - k_n\| \leq \|u_n - y_n\| + \|y_n - k_n\| \leq 2\|u_n - y_n\| \to 0.$$

Since $k_{n_i} \rightharpoonup k$, and so $u_{n_i} \rightharpoonup k$. From (3.1.17) and (3.1.26) and condition (A4) we have $\phi(y,k) \leq 0 \quad \forall y \in C$.

For any $t \in (0,1]$ and $y \in C$, let $y_t = ty + (1-t)k$,then $y_t \in C$.

By condition (A4)

$$0 = \phi(y_t, y_t) \leq t\phi(y_t, y) + (1-t)\phi(y_t, k) \leq t\phi(y_t, y).$$

Hence $\phi(y_t, y) \geq 0$. By condition (A3), $\phi(k, y) \geq 0, \forall y \in C, i.e., k \in EP(\phi)$.

The conclusion $k \in F \cap VI(C,A) \cap EP(\phi)$ is proved.

Since $z = P_{F\cap VI(C,A)\cap EP(\phi)} f(z)$, it follows from(2.2.2), (3.1.14) and (3.1.22)that

$$\begin{aligned}
\limsup_{n\to\infty} \langle f(z) - z, x_n - z\rangle &= \limsup_{n\to\infty} \langle f(z) - z, (x_n - W_n k_n) + (W_n k_n - z)\rangle \\
&\leq \limsup_{n\to\infty} \langle f(z) - z, W_n k_n - z\rangle \\
&= \lim_{n\to\infty} \langle f(z) - z, W_{n_i} k_{n_i} - z\rangle \\
&= \langle f(z) - z, k - z\rangle \leq 0. \qquad (3.1.27)
\end{aligned}$$

(VI) Finally we prove $x_n \to z (n \to \infty)$. In fact, from (3.1.27) and (3.1.3),

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \langle \alpha_n(f(x_n) - z) + \beta_n(x_n - z) + \gamma_n(W_n k_n - z), x_{n+1} - z\rangle \\
&\leq \alpha_n\langle f(x_n - f(z), x_{n+1} - z\rangle + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle + \\
&\quad \beta_n\|x_n - z\|\|x_{n+1} - z\| + \gamma_n\|k_n - z\|\|x_{n+1} - z\| \\
&\leq \frac{1}{2}\alpha_n\xi\{\|x_n - z\|^2 + \|x_{n+1} - z\|^2\} + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle + \\
&\quad \frac{1}{2}\beta_n(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \frac{1}{2}\gamma_n(\|k_n - z\|^2 + \|x_{n+1} - z\|^2) \\
&\leq \frac{1}{2}\alpha_n\xi(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\quad + \frac{1}{2}(1 - \alpha_n)(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\
&\leq \frac{1}{2}[1 - (1-\xi)\alpha_n]\|x_n - z\|^2 + \frac{1}{2}\|x_{n+1} - z\|^2 + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle,
\end{aligned}$$

which implies that

$$\|x_{n+1} - z\|^2 \leq (1 - (1-\xi)\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle.$$

Taking $a_n = \|x_n - z\|^2, \gamma_n = (1-\xi)\alpha_n, b_n = 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle$ in Lemma 2.4, by the assumptions of Theorem 3.1, we know that all condition in Lemma 2.4 are satisfied. It follows from Lemma 2.4 that $x_n \to z = P_{F\cap VI(C,A)\cap EP(\phi)}f(z)$ $(n \to \infty)$. By (3.1.17) $u_n \to z = P_{F\cap VI(C,A)\cap EP(\phi)}f(z)$ also.

This completes the proof of Theorem 3.1.

CHAPTER IV

Examples and Numerical Results

4.1 Example and Numerical Result

In this chapter, we give numerical example which support the main result.

Example 4.1.1 Let $C = [0,1]$ and the other conditions as follow :
$u_n = T_{r_n}x_n = \dfrac{x_n}{1-24r_n}, \lambda_n = \dfrac{1}{n+1}, \alpha_n = \dfrac{1}{2n}, \beta_n = \dfrac{1}{2n+1} + \dfrac{1}{2},$
$\gamma_n = \dfrac{1}{2} - \dfrac{4n+1}{4n^2+2n}, r_n = 1 - \dfrac{4n+1}{4n^2+2n}\ Ax = 2x, f(x) = \dfrac{x+2}{4}, W_n = \sin nx$

<u>**Solution**</u> We will check that the above assumptions satisfy the conditions in the main theorem.

1. $u_n = T_{r_n}x_n = \dfrac{x_n}{1-24r_n}.$

<u>Check</u>

Let $\phi(x,y)10x^2 - 14y^2 + 4xy.$

(A1.) $\phi(x,x) = 0, \forall x \in C$

$$\begin{aligned}\phi(x,x) &= 10x^2 - 14x^2 + 4x^2\\ &= 0\end{aligned}$$

(A2) $\phi(x,y) + \phi(y,x) \leq 0, \forall x,y \in C.$

$$\begin{aligned}\phi(x,y) + \phi(y,x) &= (10x^2 - 14y^2 + 4xy) + (10y^2 - 14x^2 + 4xy)\\ &= 10x^2 - 14y^2 + 4xy + 10y^2 - 14x^2 + 4xy\\ &= -4x^2 - 4y^2 + 8xy\\ &= -[(2x)^2 + 2(2x)(2y) + (2y)^2]\\ &= -(2x+2y)^2 \leq 0\end{aligned}$$

(A3) $\limsup_{t\to 0}\ \phi(tz + (1-t)x, y) \leq \phi(x,y), \quad \forall x,y,z \in C.$

$$\begin{aligned}\limsup_{t\to 0}\ \phi(tz+(1-t)x,y) &= \limsup_{t\to 0}\ 10[(tz+(1-t)x)^2 - 14y^2 + 4(tz+(1-t)x)y]\\ &= \limsup_{t\to 0}\ [10[t^2z^2 + 2(tz)(1-t)x + ((1-t)x)^2] -\\ &\quad 14y^2 + 4tzy + 4xy - 4ty]\end{aligned}$$

$$= \limsup_{t\to 0} \ [10[t^2z^2 + 2tzx - 2t^2zx + x^2 - 2tx^2 + t^2x^2] - 14y^2 + 4tzy + 4xy - 4ty]$$

$$= \limsup_{t\to 0} \ [10t^2z^2 + 20tzx - 20t^2zx + 10x^2 - 20tx^2 + 10t^2x^2] - 14y^2 + 4tzy + 4xy - 4ty]$$

$$= \limsup_{t\to 0} \ [10x^2 - 14y^2 + 4xy]$$

$$= 10x^2 - 14y^2 + 4xy = \phi(x, y).$$

(A4) $y \mapsto \phi(x, y)$ is convex and weakly lower semi-continuous.

- weakly lower semi-continuous

$$\phi(x, y_o) \leq \liminf_{n\to\infty} \ \phi(x, y_n)$$

$$\liminf_{n\to\infty} \ \phi(x, y_n) = \liminf_{n\to\infty}[10x^2 - 14y_n^2 + 4xy_n] = 10x^2 - 14y_0^2 + 4xy_0$$

so, $\liminf_{n\to\infty} \ \phi(x, y_n) = 10x^2 - 14y_0^2 + 4xy_0 = \phi(x, y_0)$

- convex if for any $x_1, x_2 \in C\text{and}t \in (0, 1)$

$$F_1(x, tx_1 + (1-t)y_1) \leq tF_1(x, x_1) + (1-t)F_1(x, y_1)$$

$$\begin{aligned}
F_1(x, tx_1 + (1-t)y_1) &= 10x^2 - 14[tx_1 + (1-t)y_1]^2 + 4x[tx_1 + (1-t)y_1] \\
&= 10x^2 - 14[tx_1 + (y_1 - ty_1)]^2 + 4x[tx_1 + (y_1 - ty_1)] \\
&= 10x^2 - 14[t^2x_1^2 + 2tx_1(y_1 - ty_1) + (y_1 - ty_1)^2] + 4txx_1 + \\
&\quad 4xy_1 - 4txy_1 \\
&= 10x^2 - 14[t^2x_1^2 + 2tx_1y_1 - 2t^2x_1y_1 + y_1^2 - 2ty_1^2 + t^2y_1^2] + \\
&\quad 4txx_1 + 4xy_1 \\
&= 10x^2 - 14t^2x_1^2 - 28tx_1y_1 + 28t^2x_1y_1 - 14y_1^2 + 28ty_1^2 - \\
&\quad 14t^2y_1^2 + 4txx_1 - 4txy_1 + 4xy_1 - 4txy_1 + 10tx^2 - 10tx^2 \\
&= 10tx^2 - 14t^2x_1^2 + 4txx_1 + 10x^2 - 14y_1^2 + 4xy_1 - 10tx^2 - \\
&\quad 14t^2y_1^2 - 4txy_1 - 28tx_1y_1 + 28t^2x_1y_1 + 28ty_1^2 \\
&= t[10x^2 - 14tx_1^2 + 4xx_1] + (1-t)10x^2 - 14y_1^2 - 14t^2y_1^2 + \\
&\quad (1-t)4xy_1 - 28tx_1y_1 + 28t^2x_1y_1 + 28ty_1^2 \\
&\leq t[10x^2 - 14x_1^2 + 4xx_1] + (1-t)[10x^2 - 14y_1^2 - 4xy_1] \\
&\quad - 28tx_1y_1 + 28t^2x_1y_1 + 28ty_1^2 \\
&\leq t[10x^2 - 14x_1^2 + 4xx_1] + (1-t)[10x^2 - 14y_1^2 - 4xy_1]
\end{aligned}$$

$$= tF_1(x,x_1) + (1-t)F_1(x,y_1).$$

Consider $\phi(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0$

$$\begin{aligned}
\phi(z,y) + \frac{1}{r}\langle y-z, z-x\rangle &= 10z^2 - 14y^2 + 4zy + \frac{1}{r}\langle y-z, z-x\rangle \\
&= 10z^2 - 14y^2 + 4zy + \frac{1}{r}(y-z)(z-x) \\
&= 10z^2 - 14y^2 + 4zy + \frac{1}{r}(yz - xy - z^2 + xz) \\
&= 10rz^2 - 14ry^2 + 4rzy + yz - xy - z^2 + xz \\
&= -14ry^2 + (4rz + z - x)y + (10rz^2 - z^2 + xz).
\end{aligned}$$

Let $a = -14r, b = 4rz + z - x, c = 10rz^2 - z^2 + xz.$

Consider $b^2 - 4ac = (4rz + z - x)^2 - 4(-14r)(10rz^2 - z^2 + xz)$

$$\begin{aligned}
&= (4rz+z)^2 - 2(4rz+z)z + x^2 + 56r(10rz^2 - z^2 + xz) \\
&= 16r^2z^2 + 8rz^2 + z^2 - 8rzx - 2zx + x^2 + 560r^2z^2 - 56rz^2 + \\
&\quad 56rxz \\
&= (16r^2z^2 + 560r^2z^2) + (8rz^2 - 56rz^2) + (56rxz - 8rzx) + \\
&\quad z^2 + x^2 - 2zx \\
&= 576r^2z^2 - 48rz^2 + 48rxz + z^2 + x^2 - 2zx \\
&= (576r^2z^2 - 48rz^2 + z^2) + (48rxz - 2xz) + x^2 \\
&= (576r^2 - 48r + 1)z^2 + (48r - 2)xz + x^2 \\
&= (24^2r^2 - 2(24)r + 1)z^2 + 2(24r - 1)xz + x^2 \\
&= [(24r-1)^2]z^2 + 2(24r-1)xz + x^2 \\
&= [(24r-1)z + x]^2 \geq 0.
\end{aligned}$$

$b^2 - 4ac = [(24r-1)z + x]^2$

$$\therefore (1 - 24r)z = x$$

$$z = \frac{x}{(1-24x)}$$

$$\therefore u_n = Tr_n x_n = \frac{x_n}{1 - 24r_n}$$

2. Let $\lambda_n = \dfrac{1}{n+1}$.

<u>Check</u>

- $\lim\limits_{n\to\infty} \left|\lambda_{n+1} - \lambda_n\right| = 0 \quad ; \ \lambda_n = \dfrac{1}{n+1}$

$$\lim_{n\to\infty} \left|\lambda_{n+1} - \lambda_n\right| = \lim_{n\to\infty} \left|\frac{1}{(n+1)+1} - \frac{1}{n+1}\right|$$

$$= \lim_{n\to\infty} \left|\frac{n+1-(n+2)}{(n+2)(n+1)}\right|$$

$$= \lim_{n\to\infty} \left|\frac{-1}{(n+2)(n+1)}\right|$$
$$= 0$$

3. Let $\alpha_n = \dfrac{1}{2n}$.

<u>Check</u>

- $\lim\limits_{n\to\infty} \alpha_n = 0 \quad , \ \sum\limits_{n=1}^{\infty} \alpha_n = \infty$

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \frac{1}{2n}$$
$$= 0$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{2n} = \infty.$$

4. Let $\beta_n = \dfrac{1}{2n+1} + \dfrac{1}{2}$.

<u>Check</u>

- $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$

$$0 < \liminf_{n\to\infty} \left[\frac{1}{2n+1} + \frac{1}{2}\right] \leq \limsup_{n\to\infty} \left[\frac{1}{2n+1} + \frac{1}{2}\right] < 1$$

5.Let $r_n = 1 - \dfrac{4n+1}{4n^2+2n}$.

<u>Check</u>

- $\liminf\limits_{n\to\infty} r_n > 0; \sum\limits_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$

$$\liminf_{n\to\infty} \left[1 - \frac{4n+1}{4n^2+2n}\right] = 1 > 0$$

$$\sum_{n+1}^{\infty} \left|1 - \frac{4(n+1)+1}{4(n+1)^2+2(n+1)} - 1 + \frac{4n+1}{4n^2+2n}\right|$$

$$= \sum_{n+1}^{\infty} \left| \frac{-4n-4+1}{4(n+1)^2+(2)(n+1)} + \frac{4n+1}{4n^2+2n} \right|$$
$$= \sum_{n+1}^{\infty} \left| \frac{-4n-3}{4(n+1)^2+2(n+1)} + \frac{4n+1}{4n^2+2n} \right| < \infty$$

6. A is and α-inverse strengly monotone.

Check

By definition of α-inverse strongly monotone

$$A : C \to H; \langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2$$

Let $Ax = 2x \quad \therefore Ay = 2y$

$$\begin{aligned} \langle Ax - Ay, x - y \rangle &= \langle 2x - 2y, x - y \rangle \\ &= (2x - 2y)(x - y) \\ &= 2(x - y)(x - y) \\ &= 2(x - y)^2 \\ &= \frac{1}{2}(2(x - y))^2 \\ &= \frac{1}{2}(2x - 2y)^2 \\ &= \frac{1}{2}\|2x - 2y\|^2. \end{aligned}$$

Hence A is a $\frac{1}{2}$ -inverse strongly monotone.

7.$f(x)$ is contraction

Check

$$\|f(x) - f(y)\| \leq \alpha \|x - y\| \quad , \alpha < 1$$

Let $f(x) = \dfrac{x+2}{4}$.

Consider : $\|f(x) - f(y)\| = \|\frac{x+2}{4} - (\frac{y+2}{4})\|$

$$\begin{aligned} &= \frac{1}{4}\|x + 2 - y - 2\| \\ &= \frac{1}{4}\|x - y\| \\ &\leq \frac{1}{2}\|x - y\|. \end{aligned}$$

Hence f is a contraction.

8.W_n is nonexpansive.

Check

$$\|Sx - Sy\| \leq \|x - y\|$$

Let $W_n = \sin nx$

consider : $\|Sx - Sy\| = \|sin\, nx - sin\, ny\|$

$$\leq \|x - y\|$$

After we run this algorithm by using MATLAB with the setting $x_1 = 0.5$ we can get the value of x_n as follows:

n	x_n	n	x_n
1	0.5000	11	0.0809
2	0.7292	12	0.0676
3	0.6810	13	0.0581
4	0.5495	14	0.0510
5	0.4155	15	0.0456
6	0.3059	16	0.0413
7	0.2245	.	.
8	0.1669	.	.
9	0.1272	.	.
10	0.0998	1001	0.0000

Table 1: The value of x_n generated by Example 4.1.1

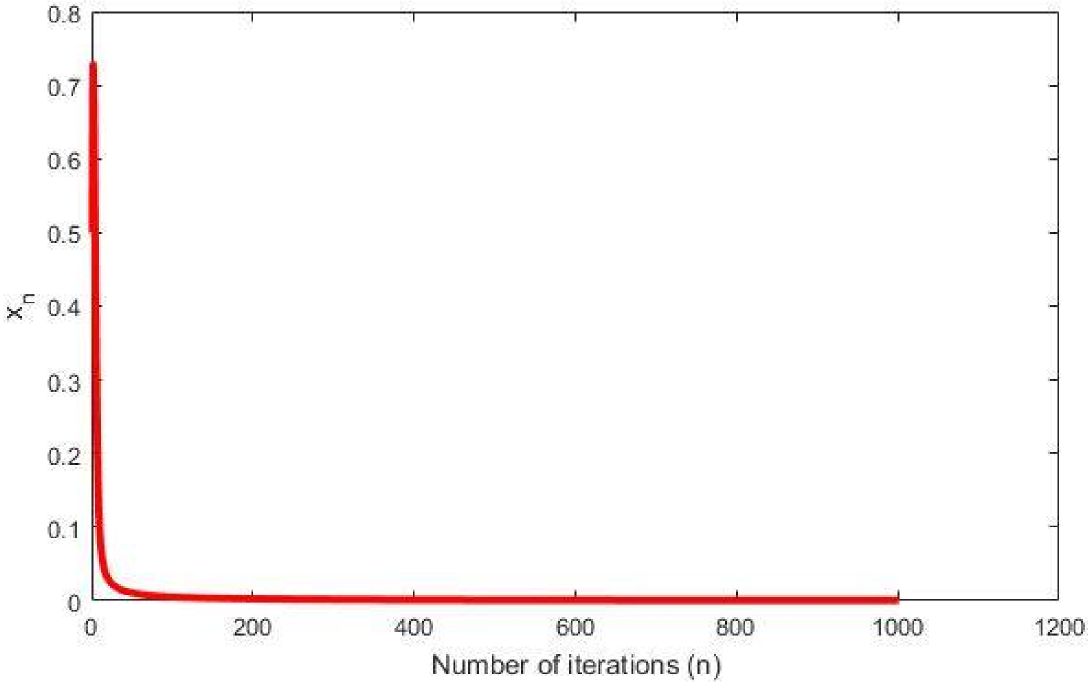

Figure 1: The convergence of x_n by Example 4.1.1

We can see that x_n converges to 0 which is the fixed point of W_n.

CHAPTER V

CONCLUSIONS

5.1 Conclusion

In this Chapter , we propose the conclusion of our study which consists of the main theorem , and numerical examples as shown in the followings.

Theorem 5.1.1 ([17]) Let H be real Hilbert space , C be a noneempty closed convex subset of H , $\phi : C \times C \to \mathbb{R}$ be a bifunction satisfying the condition (A1) - (A4) ,$A : C \to H$ an α - *inverse-strongly monotone mapping* , $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings with $F \cap VI(C, A) \cap EP(\phi) \neq \emptyset$, where $F := \bigcap_{i=1}^{\infty} F(S_i)$ and $f : H \to H$ be a contraction mapping with a contractive constant $\xi \in (0, 1)$.

Let $\{x_n\}, \{y_n\}, \{k_n\}$ and $\{u_n\}$ be the sequence defined by

$$\begin{cases} \phi(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n k_n \qquad \forall n \geq 1, \\ k_n = Pc(y_n - \lambda_n A y_n), \\ y_n = Pc(u_n - \lambda_n A u_n), \end{cases}$$

where $W_n : C \to C$ is the sequence definde by (2.2.6) ,α_n , β_n and γ_n are sequences in [0 , 1] , λ_n is a sequence in $[a, b] \subset (0, 2\alpha)$ and $\{r_n\}$ is a sequence in $\subset (0, \infty)$. If the followwing conditions are satisfied :

(i) $\alpha_n + \beta_n + \gamma_n = 1$

(ii) $\lim\limits_{n\to\infty} \alpha_n = 0$; $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$

(iii) $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$;

(iv) $\limsup\limits_{n\to\infty} r_n > 0$;$\sum\limits_{n=1}^{\infty} | r_{n+1} - r_n | < \infty$

(v) $\limsup\limits_{n\to\infty} | \lambda_{n+1} - \lambda_n | = 0$

then $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F \cap VI(C, A) \cap EP(\phi)$, where $z = P_{F\cap VI(C,A)\cap EP(\phi)} f(z)$.

Example 5.1.1 Let $C = [0, 1]$ and the other conditions as follow :

$u_n = T_{r_n} x_n = \dfrac{x_n}{1 - 24r_n}, \quad \lambda_n = \dfrac{1}{n+1}, \quad \alpha_n = \dfrac{1}{2n}, \quad \beta_n = \dfrac{1}{2n+1} + \dfrac{1}{2},$

$\gamma_n = \dfrac{1}{2} - \dfrac{4n+1}{4n^2+2n}, \quad r_n = 1 - \dfrac{4n+1}{4n^2+2n}, \ Ax = 2x, \ \ f(x) = \dfrac{x+2}{4}, W_n = \sin nx$

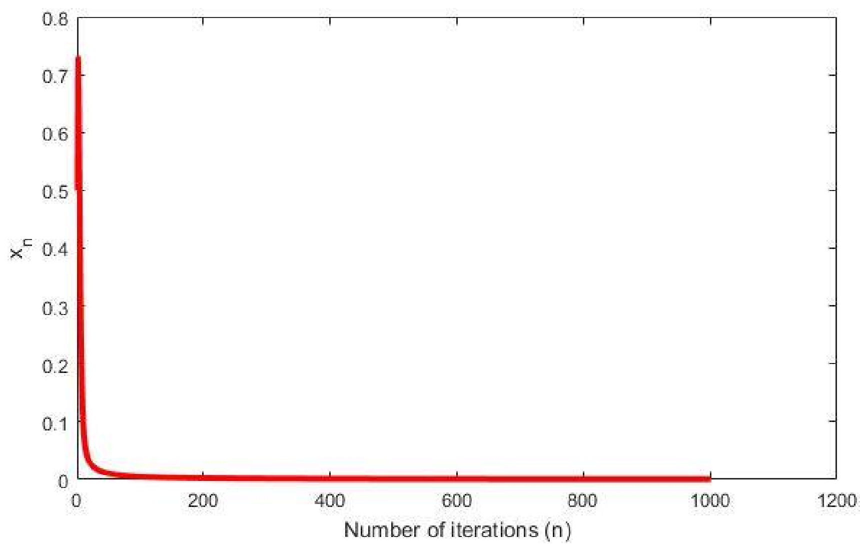

Figure 2: The convergence of x_n by Example 5.1.1

We can see that x_n converges to 0 which is the fixed point of W_n.

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