ROUGH SET THEORY APPLIED TO UP-ALGEBRAS

THEEYARAT KLINSEESOOK
SUKHONTHA BUKOK

An Independent Study Submitted in Partial Fulfillment
of the Requirements for the degree of Bachelor
of Science Program in Mathematics
April 2018
University of Phayao

ROUGH SET THEORY APPLIED TO UP-ALGEBRAS

THEEYARAT KLINSEESOOK
SUKHONTHA BUKOK

An Independent Study Submitted in Partial Fulfillment

of the Requirements for the degree of Bachelor

of Science Program in Mathematics

April 2018

University of Phayao

Advisor and Dean of School of Science have considered the independent study entitled "Rough Set Theory applied to UP-Algebras" submitted in partial fulfillment of the requirements for the degree of Bachelor of Science Program in Mathematics is hereby approved

....

(Assoc. Prof. Dr. Tanakit Thianwan)

Chairman

....

(Assist. Prof. Dr. Aiyared Iampan)

Committee and Advisor

....

(Lect. Peerapong Suebsan)

Committee

....

(Assoc. Prof. Preeyanan Sanpote)

Dean of School of Science

April 2018

ACKNOWLEDGEMENTS

First of all, we would like to express my sincere gratitude to our advisor, Assistant Professor Dr. Aiyared Iampan, who gave invaluable help, guidance and constant encouragement in all the time of independent study and writing of this report.

In addition, we would like to thank for all lectures in Department of Mathematics, University of Phayao, who opened the door of knowledge to us.

Finally, we most gratefully acknowledge our parents and my friends for all their support throughout the period of this independent study.

Theeyarat Klinseesook

Sukhontha Bukok

ชื่อเรื่อง	การนำทฤษฎีเซตหยาบมาใช้กับพีชคณิตยูพี
ผู้ศึกษาค้นคว้า	นางสาวทียารัตน์ กลิ่นสีสุข
	นางสาวสุคนธา บูโกก
อาจารย์ที่ปรึกษา	ผู้ช่วยศาสตราจารย์ ดร.อัยเรศ เอี่ยมพันธ์
วิทยาศาสตรบัณฑิต	สาขาวิชาคณิตศาสตร์
คำสำคัญ	พีชคณิตยูพี พีชคณิตยูพีย่อยหยาบ ตัวกรองยูพีหยาบ ไอดีลยูพีหยาบ ไอดีลยูพีอย่างเข้มหยาบ

บทคัดย่อ

ในงานวิจัยนี้ ได้นำทฤษฎีเซตหยาบมาใช้ในพีชคณิตยูพี อีกทั้งยังได้ศึกษาแนวคิดของพีชคณิตยูพีย่อยหยาบ ตัวกรองยูพีหยาบ ไอดีลยูพีหยาบ และไอดีลยูพีอย่างเข้มหยาบ และพิสูจน์การวางนัยทั่วไปของแนวคิดข้างต้น นอกจากนี้ ยังได้ศึกษาความสัมพันธ์ระหว่างพีชคณิตย่อยยูพีหยาบ (ตัวกรองยูพีหยาบ ไอดีลยูพีหยาบ และไอดีลยูพีอย่างเข้มหยาบ ตามลำดับ) และพีชคณิตย่อยยูพี (ตัวกรองยูพี ไอดีลยูพี และไอดีลยูพีอย่างเข้ม ตามลำดับ) และนำเสนอบางตัวอย่าง

Title	Rough Set Theory applied to UP-Algebras
Author	Miss Theeyarat Klinseesook
	Miss Sukhontha Bukok
Advisor	Assistant Professor Dr. Aiyared Iampan
Bachelor of Science	Program in Mathematics
Keywords	UP-algebra, rough UP-subalgebra, rough UP-filter,
	rough UP-ideal, rough strongly UP-ideal

ABSTRACT

In this paper, rough set theory is applied to UP-algebras, proved some results and discussed the generalization of some notions of rough UP-subalgebras, rough UP-filters, rough UP-ideals and rough strongly UP-ideals. Furthermore, we discuss the relation between rough UP-subalgebras (resp., rough UP-filters, rough UP-ideals and rough strongly UP-ideals) and UP-subalgebras (resp., UP-filters, UP-ideals and strongly UP-ideals) and present some examples.

LIST OF CONTENTS

	Page
Approved	i
Acknowledgements	ii
Abstract in Thai	iii
Abstract in English	iv
List of Contents	v
List of Figures	vii
Chapter 1 Introduction	1
Chapter 2 Basic Results on UP-Algebras	2
Chapter 3 Rough UP-Algebras	4
Chapter 4 Main Results	8
Chapter 5 Conclusions	20
Bibliography	22

Appendix

Research Manuscript 26

Biography 39

LIST OF FIGURES

Figures **Page**

1 Relation 1 13

2 Relation 2 19

CHAPTER 1
Introduction

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [6], BCI-algebras [7], BCH-algebras [4], KU-algebras [13], SU-algebras [9], UP-algebras [5] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [7] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [6, 7] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

The notion of rough sets was first considered by Pawlak [12] in 1982. After the introduction of the notion of rough sets, several authors were conducted on the generalizations of the notion of rough sets and application to many many algebraic structures such as: In 1994, Biswas and Nanda [1] introduced and discussed the notion of rough groups and rough subgroups. Rough set theory is applied to semigroups and groups by Kuroki [10], and Kuroki and Mordeson [11] in 1997. In 2002, Jun [8] and Dudek et al. [2] applied rough set theory to BCK-algebras and BCI-algebras. In 2016, Mao and Zhou [8] applied rough set theory to pseudo-BCK-algebras.

In this paper, we apply the rough set theory to UP-algebras, introduce the notion of upper and lower rough UP-subalgebras (resp., rough UP-filters, rough UP-ideals and rough strongly UP-ideals) of UP-algebras, and discuss some of their important properties and its generalizations.

CHAPTER 2
Basic Results on UP-Algebras

An algebra $X = (X, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* [5], where X is a nonempty set, $\cdot$ is a binary operation on X, and 0 is a fixed element of X (i.e., a nullary operation) if it satisfies the following axioms: for any $x, y, z \in X$,

(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,

(UP-2) $0 \cdot x = x$,

(UP-3) $x \cdot 0 = 0$, and

(UP-4) $x \cdot y = 0$ and $y \cdot x = 0$ imply $x = y$.

From [5], we know that the notion of UP-algebras is a generalization of KU-algebras.

Example 2.1. [5] Let X be a universal set. Define two binary operations $\cdot$ and $*$ on the power set of X by putting $A \cdot B = B \cap A'$ and $A * B = B \cup A'$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X), \cdot, \emptyset)$ and $(\mathcal{P}(X), *, X)$ are UP-algebras and we shall call it the *power UP-algebra of type 1* and the *power UP-algebra of type 2*, respectively.

The following is an important property of UP-algebras.

Proposition 2.2. *[5] In a UP-algebra X, the following properties hold: for any $x, y, z \in X$,*

(1) $x \cdot x = 0$,

(2) $x \cdot y = 0$ *and* $y \cdot z = 0$ *imply* $x \cdot z = 0$,

(3) $x \cdot y = 0$ *implies* $(z \cdot x) \cdot (z \cdot y) = 0$,

(4) $x \cdot y = 0$ *implies* $(y \cdot z) \cdot (x \cdot z) = 0$,

(5) $x \cdot (y \cdot x) = 0$,

(6) $(y \cdot x) \cdot x = 0$ *if and only if* $x = y \cdot x$, *and*

(7) $x \cdot (y \cdot y) = 0$.

In what follows, let X denote a UP-algebra unless otherwise specified.

Definition 2.3. [5] A subset S of X is called a *UP-subalgebra* of X if the constant 0 of X is in S, and $(S, \cdot, 0)$ itself forms a UP-algebra.

Iampan [5] proved the useful criteria that a nonempty subset S of a UP-algebra $X = (X, \cdot, 0)$ is a UP-subalgebra of X if and only if S is closed under the $\cdot$ multiplication on X.

Definition 2.4. [14] A subset F of X is called a *UP-filter* of X if it satisfies the following properties:

(1) the constant 0 of X is in F, and

(2) for any $x, y \in X, x \cdot y \in F$ and $x \in F$ imply $y \in F$.

Definition 2.5. [5] A subset B of X is called a *UP-ideal* of X if it satisfies the following properties:

(1) the constant 0 of X is in B, and

(2) for any $x, y, z \in X, x \cdot (y \cdot z) \in B$ and $y \in B$ imply $x \cdot z \in B$.

Definition 2.6. [3] A subset C of X is called a *strongly UP-ideal* of X if it satisfies the following properties:

(1) the constant 0 of X is in C, and

(2) for any $x, y, z \in X, (z \cdot y) \cdot (z \cdot x) \in C$ and $y \in C$ imply $x \in C$.

Theorem 2.7. *[3] The following statements hold:*

(1) every UP-filter of X *is a UP-subalgebra,*

(2) every UP-ideal of X *is a UP-filter, and*

(3) every strongly UP-ideal of X *is a UP-ideal. Moreover, a UP-algebra* X *is the only one strongly UP-ideal of itself.*

CHAPTER 3
Rough UP-Algebras

Definition 3.1. Let X be a set and ρ an equivalence relation on X and let $\mathcal{P}(X)$ denote the power set of X. If $x \in X$, then the ρ-class of x is the set $(x)_\rho$ defined as follows:

$$(x)_\rho = \{y \in X \mid (x, y) \in \rho\}.$$

Define the functions ρ_- and ρ_+ from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ putting for every $S \in \mathcal{P}(X)$,

$$\rho_-(S) = \{x \in X \mid (x)_\rho \subseteq S\},$$
$$\rho_+(S) = \{x \in X \mid (x)_\rho \cap S \neq \emptyset\}.$$

$\rho_-(S)$ is called the *lower approximation of* S while $\rho_+(S)$ is called the *upper approximation of* S. The set S is called *definable* if $\rho_-(S) = \rho_+(S)$ and *rough* otherwise. The pair (X, ρ) is called an *approximation space*.

Proposition 3.2. *Let A and B be nonempty subsets of a UP-algebra X. If ρ is an equivalence relation on X, then the following statements hold:*

(1) $\rho_-(A) \subseteq A \subseteq \rho_+(A)$,

(2) $A \subseteq B$ *implies* $\rho_-(A) \subseteq \rho_-(B)$ *and* $\rho_+(A) \subseteq \rho_+(B)$,

(3) $\rho_-(A \cap B) = \rho_-(A) \cap \rho_-(B)$,

(4) $\rho_-(A \cup B) \supseteq \rho_-(A) \cup \rho_-(B)$,

(5) $\rho_+(A \cap B) \subseteq \rho_+(A) \cap \rho_+(B)$,

(6) $\rho_+(A \cup B) = \rho_+(A) \cup \rho_+(B)$,

(7) $\rho_-(A') \subseteq (\rho_-(A))'$,

(8) $(\rho_+(A))' \subseteq \rho_+(A')$, *and*

(9) $\rho_-(A - B) \subseteq \rho_-(A) - \rho_-(B)$.

Proof. (1) Let $x \in \rho_-(A)$. Then $(x)_\rho \subseteq A$. By reflexivity, $(x, x) \in \rho$ so $x \in (x)_\rho$. Thus $x \in A$, that is, $\rho_-(A) \subseteq A$. Let $y \in A$. By reflexivity, $(y, y) \in \rho$ so $y \in (y)_\rho$. Thus $y \in (y)_\rho \cap A \neq \emptyset$. So $y \in \rho_+(A)$, that is, $A \subseteq \rho_+(A)$. Therefore, $\rho_-(A) \subseteq A \subseteq \rho_+(A)$.

(2) Assume that $A \subseteq B$. Let $x \in \rho_-(A)$. Then $(x)_\rho \subseteq A \subseteq B$. Thus $x \in \rho_-(B)$, that is, $\rho_-(A) \subseteq \rho_-(B)$. Let $x \in \rho_+(A)$. Then $(x)_\rho \cap A \neq \emptyset$, so there is $y \in (x)_\rho \cap A$. Thus $y \in (x)_\rho$ and $y \in A \subseteq B$, that is, $y \in (x)_\rho \cap B \neq \emptyset$. Thus $x \in \rho_+(B)$. Hence, $\rho_+(A) \subseteq \rho_+(B)$.

(3) By Proposition 3.2 (2), we get $\rho_-(A \cap B) \subseteq \rho_-(A)$ and $\rho_-(A \cap B) \subseteq \rho_-(B)$. Hence, $\rho_-(A \cap B) \subseteq \rho_-(A) \cap \rho_-(B)$. On the other hand, let $x \in \rho_-(A) \cap \rho_-(B)$. Then $x \in \rho_-(A)$ and $x \in \rho_-(B)$. Thus $(x)_\rho \subseteq A$ and $(x)_\rho \subseteq B$. So $(x)_\rho \subseteq A \cap B$, that is, $x \in \rho_-(A \cap B)$. Therefore, $\rho_-(A) \cap \rho_-(B) \subseteq \rho_-(A \cap B)$. Hence, $\rho_-(A) \cap \rho_-(B) = \rho_-(A \cap B)$.

(4) By Proposition 3.2 (2), we get $\rho_-(A) \subseteq \rho_-(A \cup B)$ and $\rho_-(B) \subseteq \rho_-(A \cup B)$. Hence, $\rho_-(A) \cup \rho_-(B) \subseteq \rho_-(A \cup B)$.

(5) By Proposition 3.2 (2), we get $\rho_+(A \cap B) \subseteq \rho_+(A)$ and $\rho_+(A \cap B) \subseteq \rho_+(B)$. Hence, $\rho_+(A \cap B) \subseteq \rho_+(A) \cap \rho_+(B)$.

(6) Let $x \in \rho_+(A \cup B)$. Then $(x)_\rho \cap (A \cup B) \neq \emptyset$. Thus $((x)_\rho \cap A) \cup ((x)_\rho \cap B) \neq \emptyset$, we have $(x)_\rho \cap A \neq \emptyset$ or $(x)_\rho \cap B \neq \emptyset$. Hence, $x \in \rho_+(A)$ or $x \in \rho_+(B)$. Therefore, $x \in \rho_+(A) \cup \rho_+(B)$, that is, $\rho_+(A \cup B) \subseteq \rho_+(A) \cup \rho_+(B)$. On the other hand, $\rho_+(A) \subseteq \rho_+(A \cup B)$ and $\rho_+(B) \subseteq \rho_+(A \cup B)$ by Proposition 3.2 (2). Hence, $\rho_+(A) \cup \rho_+(B) \subseteq \rho_+(A \cup B)$, that is, $\rho_+(A \cup B) = \rho_+(A) \cup \rho_+(B)$.

(7) Let $x \in \rho_-(A')$. Then $(x)_\rho \subseteq A'$ and so $(x)_\rho \nsubseteq A$. Thus $x \notin \rho_-(A)$, that is, $x \in (\rho_-(A))'$. Hence, $\rho_-(A') \subseteq (\rho_-(A))'$.

(8) Let $x \in (\rho_+(A))'$. Then $x \notin \rho_+(A)$ and so $(x)_\rho \cap A = \emptyset$. Thus $x \notin A$, that is, $x \in A'$. Therefore, $(x)_\rho \cap A' \neq \emptyset$, that is, $x \in \rho_+(A')$. Hence, $(\rho_+(A))' \subseteq \rho_+(A')$.

(9) Now,

$$\begin{aligned}\rho_-(A-B) &= \rho_-(A\cap B') \\ &= \rho_-(A)\cap\rho_-(B') && ((3)) \\ &\subseteq \rho_-(A)\cap(\rho_-(B))' && ((7)) \\ &= \rho_-(A)-\rho_-(B).\end{aligned}$$

Hence, $\rho_-(A-B)\subseteq\rho_-(A)-\rho_-(B)$. □

Remark 3.3. Let ρ be an equivalence relation on a set X. Then $\rho_-(X)=X=\rho_+(X)$.

Proof. By Proposition 3.2 (1), we have $\rho_-(X)\subseteq X\subseteq\rho_+(X)$ and $\rho_+(X)\subseteq X$. Thus $X=\rho_+(X)$. We shall show that $X\subseteq\rho_-(X)$. Let $x\in X$. Then $(x)_\rho\subseteq X$. Thus $x\in\rho_-(X)$, that is, $X\subseteq\rho_-(X)$. Hence, $\rho_-(X)=X=\rho_+(X)$. □

Definition 3.4. Let ρ be a congruence relation on X. Then the set of all ρ-classes is called the *quotient set* of X by ρ, and is denoted by X/ρ. That is,

$$X/\rho=\{(x)_\rho\mid x\in X\}.$$

Define a binary operation $*$ on X/ρ by $(x)_\rho*(y)_\rho=(x\cdot y)_\rho$ for all $x,y\in X$. Then $(X/\rho,*,(0)_\rho)$ is an algebra of type $(2,0)$. Indeed, let $(x_1)_\rho=(x_2)_\rho$ and $(y_1)_\rho=(y_2)_\rho$. Then $(x_1,x_2)\in\rho$ and $(y_1,y_2)\in\rho$, so $(x_1\cdot y_1,x_2\cdot y_2)\in\rho$ because ρ is a congruence relation on X. Hence, $(x_1)_\rho*(y_1)_\rho=(x_1\cdot y_1)_\rho=(x_2\cdot y_2)_\rho=(x_2)_\rho*(y_2)_\rho$.

Definition 3.5. For nonempty subsets A and B of a UP-algebra $X=(X,\cdot,0)$, we denote

$$A\cdot B=\{a\cdot b\mid a\in A \text{ and } b\in B\}.$$

Lemma 3.6. *If ρ is a congruence relation on X, then $(x)_\rho\cdot(y)_\rho\subseteq(x\cdot y)_\rho$ for all $x,y\in X$.*

Proof. Let $x,y\in X$ and $t\in(x)_\rho\cdot(y)_\rho$. Then $t=a\cdot b$ for some $a\in(x)_\rho$ and $b\in(y)_\rho$. Thus $(a,x)\in\rho$ and $(b,y)\in\rho$. So $(a\cdot b,x\cdot y)\in\rho$, that is, $t=a\cdot b\in(x\cdot y)_\rho$. Therefore, $(x)_\rho\cdot(y)_\rho\subseteq(x\cdot y)_\rho$. □

Example 3.7. Let $X = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

Then $(X, \cdot, 0)$ is a UP-algebra. We see that

$$\rho = \{(0, 0), (1, 1), (2, 2), (3, 3), (0, 1), (1, 0)\}$$

is a congruence relation on X. Thus

$$(0)_\rho = \{0, 1\}, (1)_\rho = \{0, 1\}, (2)_\rho = \{2\}, \text{ and } (3)_\rho = \{3\}.$$

Since $(2 \cdot 2)_\rho = (0)_\rho = \{0, 1\}$ and $(2)_\rho \cdot (2)_\rho = \{2\} \cdot \{2\} = \{0\}$, we have $(2)_\rho \cdot (2)_\rho = \{0\} \not\supseteq \{0, 1\} = (2 \cdot 2)_\rho$.

Proposition 3.8. *Let A and B be nonempty subsets of X. If ρ is a congruence relation on X, then $\rho_+(A) \cdot \rho_+(B) \subseteq \rho_+(A \cdot B)$.*

Proof. Let $t \in \rho_+(A) \cdot \rho_+(B)$. Then $t = x \cdot y$ for some $x \in \rho_+(A)$ and $y \in \rho_+(B)$. Thus $(x)_\rho \cap A \neq \emptyset$ and $(y)_\rho \cap B \neq \emptyset$, that is, $a \in (x)_\rho \cap A$ and $b \in (y)_\rho \cap B$ for some $a, b \in X$. By Lemma 3.6, we have $a \cdot b \in (x)_\rho \cdot (y)_\rho \subseteq (x \cdot y)_\rho$ and $a \cdot b \in A \cdot B$, so $a \cdot b \in (x \cdot y)_\rho \cap (A \cdot B) \neq \emptyset$. Thus $(t)_\rho \cap (A \cdot B) = (x \cdot y)_\rho \cap (A \cdot B) \neq \emptyset$, that is, $t \in \rho_+(A \cdot B)$. Hence, $\rho_+(A) \cdot \rho_+(B) \subseteq \rho_+(A \cdot B)$. $\square$

Example 3.9. From Example 3.7, let $A = \{3\}$ and $B = \{2, 3\}$. Then $A \cdot B = \{0, 2\}$, $\rho_+(A) = \{3\}$ and $\rho_+(B) = \{2, 3\}$. Thus $\rho_+(A) \cdot \rho_+(B) = \{0, 2\} \not\supseteq \{0, 1, 2\} = \rho_+(A \cdot B)$.

CHAPTER 4
Main Results

In this chapter, we will research and analysis upper and lower rough UP-subalgebras (resp., rough UP-filters, rough UP-ideals and rough strongly UP-ideals) of UP-algebras, and discuss some of their important properties and its generalizations.

Definition 4.1. Let S be a nonempty subset of X and ρ an equivalence relation on X. Then S is called

(1) an *upper rough UP-subalgebra* of X if $\rho_+(S)$ is a UP-subalgebra of X,

(2) an *upper rough UP-filter* of X if $\rho_+(S)$ is a UP-filter of X,

(3) an *upper rough UP-ideal* of X if $\rho_+(S)$ is a UP-ideal of X,

(4) an *upper rough strongly UP-ideal* of X if $\rho_+(S)$ is a strongly UP-ideal of X,

(5) a *lower rough UP-subalgebra* of X if $\rho_-(S)$ is a UP-subalgebra of X when $\rho_-(S)$ is nonempty,

(6) a *lower rough UP-filter* of X if $\rho_-(S)$ is a UP-filter of X when $\rho_-(S)$ is nonempty,

(7) a *lower rough UP-ideal* of X if $\rho_-(S)$ is a UP-ideal of X when $\rho_-(S)$ is nonempty,

(8) a *lower rough strongly UP-ideal* of X if $\rho_-(S)$ is a strongly UP-ideal of X when $\rho_-(S)$ is nonempty,

(9) a *rough UP-subalgebra* of X if it is both an upper and a lower rough UP-subalgebra of X,

(10) a *rough UP-filter* of X if it is both an upper and a lower rough UP-filter of X,

(11) a *rough UP-ideal* of X if it is both an upper and a lower rough UP-ideal of X, and

(12) a *rough strongly UP-ideal* of X if it is both an upper and a lower rough strongly UP-ideal of X.

Example 4.2. Let $X = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	2
2	0	1	0	3	1
3	0	1	2	0	4
4	0	0	0	3	0

Then $(X, \cdot, 0)$ is a UP-algebra. We see that

$$\rho = \{(0,0), (1,1), (2,2), (3,3), (4,4), (0,2), (2,0), (1,4), (4,1)\}$$

is a congruence relation on X. Thus

$$(0)_\rho = (2)_\rho = \{0, 2\}, (3)_\rho = \{3\}, \text{ and } (1)_\rho = (4)_\rho = \{1, 4\}.$$

We have

(1) $S := \{0, 3\}$ is a UP-ideal (resp., UP-filter and UP-subalgebra) of X but $\rho_-(S) = \{3\}$ is not a UP-ideal (resp., UP-filter and UP-subalgebra) of X. Thus S is not a lower rough UP-ideal (resp., lower rough UP-filter and lower rough UP-subalgebra) of X. Hence, S is not a rough UP-ideal (resp., rough UP-filter and rough UP-subalgebra) of X.

(2) $S := \{0, 2, 4\}$ is not a UP-subalgebra (resp., UP-filter and UP-ideal) of X but $\rho_-(S) = \{0, 2\}$ is a UP-subalgebra (resp., UP-filter and UP-ideal) and $\rho_+(S) = \{0, 1, 2, 4\}$ is a UP-subalgebra (resp., UP-filter and UP-ideal) of X. Thus S is both a lower and an upper rough UP-subalgebra (resp., rough UP-filter and rough UP-ideal) of X. Hence, S is a rough UP-subalgebra (resp., rough UP-filter and rough UP-ideal) of X.

(3) $S := \{0, 1\}$ is a UP-ideal (resp., UP-filter and UP-subalgebra) of X. Then $\rho_-(S) = \emptyset$ and $\rho_+(S) = \{0, 1, 2, 4\}$. Thus S is both a lower and an upper rough UP-ideal (resp., rough UP-filter and rough UP-subalgebra) of X. Hence, S is a rough UP-ideal (resp., rough UP-filter and rough UP-subalgebra) of X.

(4) If $\rho = X \times X$, then $(0)_\rho = (1)_\rho = (2)_\rho = (3)_\rho = X$. Thus $S := \{1, 3\}$ is not a UP-ideal (resp., UP-filter and UP-subalgebra) of X, and $\rho_-(S) = \emptyset$ and $\rho_+(S) = X$, that is, S is both a lower and an upper rough UP-ideal of X. Hence, S is a rough UP-ideal (resp., rough UP-filter and rough UP-subalgebra) of X.

Theorem 4.3. *Let ρ be a congruence relation on X. If C is a strongly UP-ideal of X, then C is a rough strongly UP-ideal of X.*

Proof. Assume that C is a strongly UP-ideal of X. By Theorem 2.7 (3), we have $C = X$. By Remark 3.3, we have $\rho_-(C) = X = \rho_+(C)$. By Theorem 2.7 (3) again, we have $\rho_-(C)$ and $\rho_+(C)$ are strongly UP-ideals of X. Therefore, C is a rough strongly UP-ideal of X. □

Example 4.4. From Example 4.2 (4), we have $C := \{0, 1, 2\}$ is not a strongly UP-ideal of X. Since $\rho_-(C) = \emptyset$ and $\rho_+(C) = X$, we have C is both a lower and an upper rough strongly UP-ideal of X. Hence, C is a rough strongly UP-ideal of X.

Theorem 4.5. *Every rough strongly UP-ideal of X is a rough UP-ideal.*

Proof. Let S be a rough strongly UP-ideal of X. Then $\rho_-(S)$ (if $\rho_-(S)$ is nonempty) and $\rho_+(S)$ are strongly UP-ideals of X. By Theorem 2.7 (3), $\rho_-(S)$ (if $\rho_-(S)$ is nonempty) and $\rho_+(S)$ are UP-ideals of X. Hence, S is a rough UP-ideal of X. □

Example 4.6. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	0	0	2	3	2	5
2	0	1	0	3	1	5
3	0	1	2	0	4	5
4	0	0	0	3	0	5
5	0	0	2	0	2	0

Then $(X, \cdot, 0)$ is a UP-algebra. We see that

$$\rho = \{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5),(0,2),(2,0),(1,4),(4,1)\}$$

is a congruence relation on X. Thus

$$(0)_\rho = (2)_\rho = \{0,2\},\ (1)_\rho = (4)_\rho = \{1,4\},\ (3)_\rho = \{3\}, \text{ and } (5)_\rho = \{5\}.$$

If $S = \{0,2,4\}$, then $\rho_-(S) = \{0,2\}$ and $\rho_+(S) = \{0,1,2,4\}$. Thus $\rho_-(S)$ and $\rho_+(S)$ are UP-ideals of X. Hence, S is a rough UP-ideal of X. Since $\rho_-(S) \neq X$ and $\rho_+(S) \neq X$, it follows from Theorem 2.7 (3) that $\rho_-(S)$ and $\rho_+(S)$ are not a strongly UP-ideal of X. Hence, S is a rough UP-ideal of X but is not a rough strongly UP-ideal.

Theorem 4.7. *Every rough UP-ideal of X is a rough UP-filter.*

Proof. Let S be a rough UP-ideal of X. Then $\rho_-(S)$ (if $\rho_-(S)$ is nonempty) and $\rho_+(S)$ are UP-ideals of X. By Theorem 2.7 (2), we have $\rho_-(S)$ (if $\rho_-(S)$ is nonempty) and $\rho_+(S)$ are UP-filters of X. Hence, S is a rough UP-filter of X. □

Example 4.8. Let $X = \{0,1,2,3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	2
3	0	1	0	0

Then $(X, \cdot, 0)$ is a UP-algebra. We see that

$$\rho = \{(0,0),(1,1),(2,2),(3,3)\}$$

is a congruence relation on X. Thus

$$(0)_\rho = \{0\}, (1)_\rho = \{1\}, (2)_\rho = \{2\}, \text{ and } (3)_\rho = \{3\}.$$

If $S = \{0,1\}$, then $\rho_-(S) = \{0,1\} = \rho_+(S)$. Thus $\rho_-(S)$ and $\rho_+(S)$ are UP-filters of X but are not a UP-ideal. Hence, S is a rough UP-filter of X but is not a rough UP-ideal.

Theorem 4.9. *Every rough UP-filter of X is a rough UP-subalgebra.*

Proof. Let S be a rough UP-filter of X. Then $\rho_-(S)$ (if $\rho_-(S)$ is nonempty) and $\rho_+(S)$ are UP-filters of X. By Theorem 2.7 (1), we have $\rho_-(S)$ (if $\rho_-(S)$ is nonempty) and $\rho_+(S)$ are UP-subalgebras of X. Hence, S is a rough UP-subalgebra of X. $\square$

Example 4.10. From Example 4.6, if $S = \{0,1,2,5\}$, then $\rho_-(S) = \{0,2,5\}$ and $\rho_+(S) = \{0,1,2,4,5\}$. Thus $\rho_-(S)$ and $\rho_+(S)$ are UP-subalgebras of X but are not a UP-filter. Hence, S is a rough UP-subalgebra of X but is not a rough UP-filter.

By Theorem 4.5, 4.7, and 4.9 and Example 4.6, 4.8 and 4.10, we have that the notion of rough UP-subalgebras is a generalization of rough UP-filters, the notion of rough UP-filters is a generalization of rough UP-ideals, and the notion of rough UP-ideals is a generalization of rough strongly UP-ideals. By Example 4.2, the notions of UP-subalgebras (resp., UP-filters and UP-ideals) and rough UP-subalgebras (resp., rough UP-filters and rough UP-ideals) are not identical. Hence, we have the following relation:

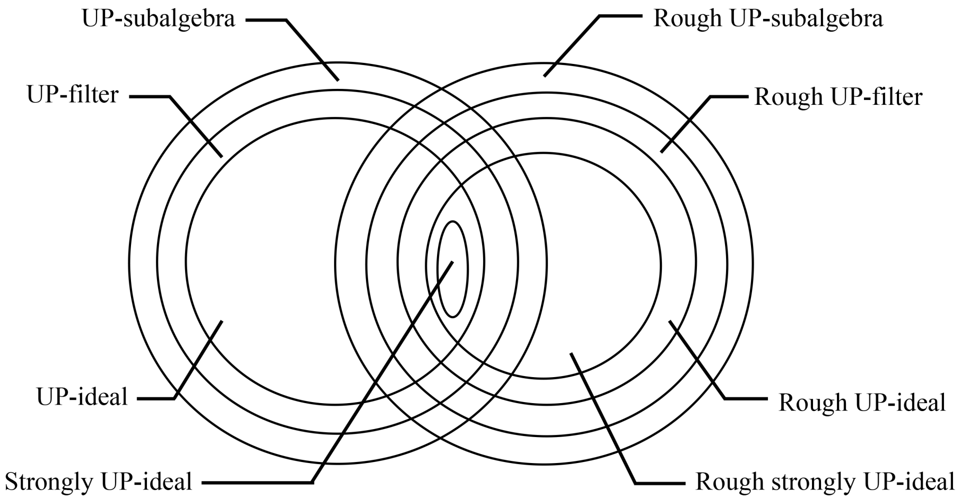


Figure 1: Relation 1

Lemma 4.11. *Let ρ be a congruence relation on X. If S is a UP-filter of X such that $(0)_\rho \subseteq S$, then $(s)_\rho \subseteq S$ for all $s \in S$.*

Proof. Assume that $(s)_\rho \nsubseteq S$ for some $s \in S$. Then there is $x \in (s)_\rho$ but $x \notin S$, so $(x, s) \in \rho$. Since ρ is a congruence relation on X, we have $(s \cdot x, 0) = (s \cdot x, s \cdot s) \in \rho$. that is, $s \cdot x \in (s \cdot x)_\rho = (0)_\rho \subseteq S$. Since S is a UP-filter of X, we have $x \in S$ which is a contradiction. Hence $(s)_\rho \subseteq S$ for all $s \in S$. $\square$

Definition 4.12. Let B be a UP-ideal of X. Define the binary relation $\sim_B$ on X as follows: for all $x, y \in X$,

$$x \sim_B y \text{ if and only if } x \cdot y \in B \text{ and } y \cdot x \in B. \tag{4.1}$$

$\sim_{B-}(S)$ is called the *lower approximation* of S by B while $\sim_{B+}(S)$ is called the *upper approximation* of S by B. The set S is called *definable* with respect to B if $\sim_{B-}(S) = \sim_{B+}(S)$ and *rough* with respect to B otherwise.

Iampan [5] proved that $\sim_B$ is a congruence relation on X.

Lemma 4.13. *If B and C are UP-ideals of X such that $B \subseteq C$, then $\sim_B \subseteq \sim_C$.*

Proof. Let $(x, y) \in \sim_B$. Then $x \cdot y, y \cdot x \in B \subseteq C$. Thus $(x, y) \in \sim_C$. Hence $\sim_B \subseteq \sim_C$. $\square$

Proposition 4.14. *Every nonempty subset of X is definable with respect to $\{0\}$.*

Proof. Let S be a nonempty subset of X. If $a \in (x)_{\sim_{\{0\}}}$, then $(a,x) \in \{0\}$. Thus $a \cdot x = 0 = x \cdot a$. By (UP-4), we have $a = x$. Thus $(x)_{\sim_{\{0\}}} = \{x\}$ for all $x \in X$. Now,

$$\begin{aligned}
\sim_{\{0\}_-}(S) &= \{x \in X \mid (x)_{\sim_{\{0\}}} \subseteq S\} \\
&= \{x \in X \mid \{x\} \subseteq S\} \\
&= \{x \in X \mid x \in S\} \\
&= S
\end{aligned}$$

and

$$\begin{aligned}
\sim_{\{0\}_+}(S) &= \{x \in X \mid (x)_{\sim_{\{0\}}} \cap S \neq \emptyset\} \\
&= \{x \in X \mid \{x\} \cap S \neq \emptyset\} \\
&= \{x \in X \mid x \in S\} \\
&= S.
\end{aligned}$$

Hence, $\sim_{\{0\}_-}(S) = S = \sim_{\{0\}_+}(S)$, that is, S is definable with respect to $\{0\}$. □

Theorem 4.15. *[5] Let B be a UP-ideal of X. Then the following statements hold:*

(1) the $\sim_B$-class $(0)_{\sim_B}$ is a UP-ideal and a UP-subalgebra of X which $B = (0)_{\sim_B}$,

(2) a $\sim_B$-class $(x)_{\sim_B}$ is a UP-ideal of X if and only if $x \in B$,

(3) a $\sim_B$-class $(x)_{\sim_B}$ is a UP-subalgebra of X if and only if $x \in B$, and

(4) $(X/\sim_B, *, (0)_{\sim_B})$ is a UP-algebra under the $*$ multiplication defined by $(x)_{\sim_B} * (y)_{\sim_B} = (x \cdot y)_{\sim_B}$ for all $x, y \in X$, called the quotient UP-algebra of X induced by the congruence relation $\sim_B$.

Remark 4.16. If B is a UP-ideal of X, then $\sim_{B-}(B) = B = \sim_{B+}(B)$. that is, B is definable with respect to itself.

Proof. Assume that B is a UP-ideal of X. Let $x \in B$. Then $x \cdot 0 = 0 \in B$ and $0 \cdot x = x \in B$, that is, $(x, 0) \in \sim_B$ it follows from Theorem 4.15 that $B = (0)_{\sim_B} = (x)_{\sim_B}$, so $x \in \sim_{B-}(B)$. Hence, $B \subseteq \sim_{B-}(B) \subseteq \sim_{B+}(B)$. By Proposition 3.2 (1), we have $\sim_{B-}(B) \subseteq B$ so $\sim_{B-}(B) = B$. Finally, we shall show that $\sim_{B+}(B) \subseteq B$. Let $x \in \sim_{B+}(B)$. Then $(x)_{\sim_B} \cap B \neq \emptyset$, so there is $a_x \in (x)_{\sim_B}$ and $a_x \in B = (0)_{\sim_B}$. Thus $(x)_{\sim_B} = (a_x)_{\sim_B} = (0)_{\sim_B} = B$, so $x \in B$. Thus $\sim_{B+}(B) \subseteq B$. Hence, $B = \sim_{B+}(B)$. □

Remark 4.17. Let S be a nonempty subset of X contained in a UP-ideal B of X. Then $\sim_{B+}(S) = B$ and $\sim_{B-}(S) = \emptyset$.

Proof. Let $x \in \sim_{B+}(S)$. Then $(x)_{\sim_B} \cap S \neq \emptyset$. Since $S \subset B$, we have $(x)_{\sim_B} \cap B \neq \emptyset$. By Remark 4.16, we have $x \in \sim_{B+}(B) = B$. Thus $\sim_{B+}(S) \subseteq B$. Next, we shall show that $B \subseteq \sim_{B+}(S)$. Let $x \in B$. By Theorem 4.15 (1), we have $x \in B = (0)_{\sim_B}$. Then $(x)_{\sim_B} = (0)_{\sim_B} = B$, so $(x)_{\sim_B} \cap B \neq \emptyset$. Thus $x \in \sim_{B+}(S)$. Hence, $\sim_{B+}(S) = B$. Finally, we shall show that $\sim_{B-}(S) = \emptyset$. Let $\sim_{B-}(S) \neq \emptyset$. Then there are $x \in \sim_{B-}(S)$. Thus $(x)_{\sim_B} \subseteq S \subset B$. By Theorem 4.15 (1), we have $x \in B = (0)_{\sim_B}$. Then $(x)_{\sim_B} = (0)_{\sim_B} = B$ which is a contradiction. Hence, $\sim_{B-}(S) = \emptyset$. □

By Remark 4.17, we can see that S is a rough UP-ideal (resp., rough UP-filter and rough UP-subalgebra) of X with respect to B.

Example 4.18. Let $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	0	0	2	3	2	3	6	7
2	0	1	0	3	1	5	3	7
3	0	1	2	0	4	1	2	7
4	0	0	0	3	0	3	3	7
5	0	0	2	0	2	0	2	7
6	0	1	0	0	1	1	0	7
7	0	0	0	0	0	0	0	0

Then $(X, \cdot, 0)$ is a UP-algebra. Let $B = \{0, 2\}$. Then B is a UP-ideal of X, so $\sim_B$ is a congruence relation on X. Thus $(0)_{\sim_B} = (2)_{\sim_B} = \{0, 2\}$, $(1)_{\sim_B} = (4)_{\sim_B} = \{1, 4\}$, $(3)_{\sim_B} = (6)_{\sim_B} = \{3, 6\}$, $(5)_{\sim_B} = \{5\}$, and $(7)_{\sim_B} = \{7\}$. Let $S = \{0, 1, 2, 3, 4, 5\}$. Then S is a UP-subalgebra of X but $\sim_{B-}(S) = \{0, 1, 2, 4, 5\}$ is not a UP-subalgebra of X. Thus S is not a lower rough UP-subalgebra of X. Hence, S is not a rough UP-subalgebra of X.

Theorem 4.19. *Let S be a UP-subalgebra of X containing a UP-ideal B of X. Then $\sim_{B+}(S)$ is a UP-subalgebra of X, that is, S is an upper rough UP-subalgebra of X with respect to B.*

Proof. By Proposition 3.2 (1), we have $S \subseteq \sim_{B+}(S) \neq \emptyset$. Let $x, y \in \sim_{B+}(S)$. Then $(x)_{\sim_B} \cap S \neq \emptyset$ and $(y)_{\sim_B} \cap S \neq \emptyset$. Thus there exist $a_x, a_y \in S$ such that $a_x \in (x)_{\sim_B}$ and $a_y \in (y)_{\sim_B}$. By Lemma 3.6, we have $a_x \cdot a_y \in (x)_{\sim_B} \cdot (y)_{\sim_B} \subseteq (x \cdot y)_{\sim_B}$. Since S is a UP-subalgebra of X, we have $a_x \cdot a_y \in S$. Thus $a_x \cdot a_y \in (x \cdot y)_{\sim_B} \cap S \neq \emptyset$. Hence, $x \cdot y \in \sim_{B+}(S)$, that is, $\sim_{B+}(S)$ is a UP-subalgebra of X. $\square$

Example 4.20. From Example 4.18, we have $S = \{0, 2, 4\}$ is not a UP-subalgebra of X but $\sim_{B+}(S) = \{0, 1, 2, 4\}$ is a UP-subalgebra of X, that is, S is an upper rough UP-subalgebra of X with respect to B.

Theorem 4.21. *Let S be a UP-filter of X containing a UP-ideal B of X. Then*

(1) $\sim_{B-}(S)$ is a UP-filter of X,

(2) $\sim_{B+}(S)$ is a UP-filter of X.

Moreover, S is a rough UP-filter of X with respect to B.

Proof. (1) Let $x \in (0)_{\sim_B}$. Then $(x, 0) \in \sim_B$, that is, $x = 0 \cdot x \in B \subseteq S$. Thus $(0)_{\sim_B} \subseteq S$. Hence $0 \in \sim_{B-}(S)$. Next, let $x \cdot y \in \sim_{B-}(S)$ and $x \in \sim_{B-}(S)$. Then $(x \cdot y)_{\sim_B} \subseteq S$ and $(x)_{\sim_B} \subseteq S$. Thus $x \in S$. We shall show that $y \in \sim_{B-}(S)$, that is, $(y)_{\sim_B} \subseteq S$. Let $a_y \in (y)_{\sim_B}$. Since $x \in (x)_{\sim_B}$, it follows from Lemma 3.6 that $x \cdot a_y \in (x)_{\sim_B} \cdot (y)_{\sim_B} \subseteq (x \cdot y)_{\sim_B} \subseteq S$. Thus $x \cdot a_y \in S$. Since S is a UP-filter of X, we have $a_y \in S$. Thus $(y)_{\sim_B} \subseteq S$, that is, $y \in \sim_{B-}(S)$. Hence, $\sim_{B-}(S)$ is a UP-filter of X.

(2) Since $0 \in (0)_{\sim_B}$ and $0 \in S$, we have $0 \in (0)_{\sim_B} \cap S \neq \emptyset$. Hence, $0 \in \sim_{B+}(S)$. Next, let $x \cdot y \in \sim_{B+}(S)$ and $x \in \sim_{B+}(S)$. Then $(x \cdot y)_{\sim_B} \cap S \neq \emptyset$ and $(x)_{\sim_B} \cap S \neq \emptyset$. We shall show that $y \in \sim_{B+}(S)$, that is, $(y)_{\sim_B} \cap S \neq \emptyset$. Let $u, v \in S$ be such that $u \in (x \cdot y)_{\sim_B}$ and $v \in (x)_{\sim_B}$. Thus $(u, x \cdot y) \in \sim_B$ and $(v, x) \in \sim_B$, so $u \cdot (x \cdot y) \in B \subseteq S$ and $v \cdot x \in B \subseteq S$. Since $u, v \in S$ and S is a UP-filter of X, we have $x \cdot y \in S$ and $x \in S$ and so $y \in S$. Since $y \in (y)_{\sim_B}$, we have $y \in (y)_{\sim_B} \cap S \neq \emptyset$. Thus $y \in \sim_{B+}(S)$. Hence, $\sim_{B+}(S)$ is a UP-filter of X. $\square$

Example 4.22. From Example 4.18, let $S = \{0, 2, 3\}$. Then S is not a UP-filter of X, But $\sim_{B-}(S) = \{0, 2\}$ and $\sim_{B+}(S) = \{0, 2, 3, 6\}$ are UP-filter of X, that is, S is both a lower and an upper rough UP-filter of X with respect to B. Hence, S is a rough UP-filter of X with respect to B.

Theorem 4.23. *Let S be a UP-ideal of X containing a UP-ideal B of X. Then*

(1) $\sim_{B-}(S)$ is a UP-ideal of X,

(2) $\sim_{B+}(S)$ is a UP-ideal of X.

Moreover, S is a rough UP-ideal of X with respect to B.

Proof. (1) Let $x \in (0)_{\sim_B}$. Then $(x, 0) \in \sim_B$, that is, $x = 0 \cdot x \in B \subseteq S$. Thus $(0)_{\sim_B} \subseteq S$. Hence, $0 \in \sim_{B-}(S)$. Next, let $x \cdot (y \cdot z) \in \sim_{B-}(S)$ and $y \in \sim_{B-}(S)$. Then $(x \cdot (y \cdot z))_{\sim_B} \subseteq S$ and $(y)_{\sim_B} \subseteq S$. Thus $y \in S$. We shall show that $x \cdot z \in \sim_{B-}(S)$, that is, $(x \cdot z)_{\sim_B} \subseteq S$. Since $x \cdot (y \cdot z) \in (x \cdot (y \cdot z))_{\sim_B} \subseteq S$ and S is a UP-ideal of X, we have $x \cdot z \in S$. By Lemma 4.11, we have $(x \cdot z)_{\sim_B} \subseteq S$. Thus $x \cdot z \in \sim_{B-}(S)$. Hence, $\sim_{B-}(S)$ is a UP-ideal of X.

(2) Since $0 \in (0)_{\sim_B}$ and $0 \in S$, we have $0 \in (0)_{\sim_B} \cap S \neq \emptyset$. Hence, $0 \in \sim_{B+}(S)$. Next, let $x \cdot (y \cdot z) \in \sim_{B+}(S)$ and $y \in \sim_{B+}(S)$. Then $(x \cdot (y \cdot z))_{\sim_B} \cap S \neq \emptyset$ and $(y)_{\sim_B} \cap S \neq \emptyset$. We shall show that $x \cdot z \in \sim_{B+}(S)$, that is, $(x \cdot z)_{\sim_B} \cap S \neq \emptyset$. Since $(x \cdot (y \cdot z))_{\sim_B} \cap S \neq \emptyset$ and $(y)_{\sim_B} \cap S \neq \emptyset$ so we have $s_1, s_2 \in S$ such that $s_1 \in (x \cdot (y \cdot z))_{\sim_B}$ and $s_2 \in (y)_{\sim_B}$. Thus $(s_1, x \cdot (y \cdot z)) \in \sim_B$ and $(s_2, y) \in \sim_B$. so $(x \cdot (s_2 \cdot z), x \cdot (y \cdot z)) \in \sim_B$. By transitive, $(s_1, x \cdot (s_2 \cdot z)) \in \sim_B$.

Thus $(s_1)_{\sim_B} = (x \cdot (s_2 \cdot z))_{\sim_B}$. Since S is a UP-ideal of X, it follows from Theorem 2.7 (2) that S is a UP-filter of X. By Lemma 4.11, we have $(s_1)_{\sim_B} \subseteq S$. Thus $(x \cdot (s_2 \cdot z))_{\sim_B} \subseteq S$. Since $x \cdot (s_2 \cdot z) \in (x \cdot (s_2 \cdot z))_{\sim_B} \subseteq S$ and S is a UP-ideal of X, we have $x \cdot z \in S$. Thus $x \cdot z \in (x \cdot z)_{\sim_B} \cap S \neq \emptyset$, that is, $x \cdot z \in \sim_{B+}(S)$. Hence, $\sim_{B+}(S)$ is a UP-ideal of X. $\square$

Example 4.24. From Example 4.18, let $S = \{0, 2, 6\}$. Then S is not a UP-ideal of X, But $\sim_{B-}(S) = \{0, 2\}$ and $\sim_{B+}(S) = \{0, 2, 3, 6\}$ are UP-ideal of X, that is, S is both a lower and an upper rough UP-ideal of X with respect to B. Hence, S is a rough UP-ideal of X with respect to B.

Theorem 4.25. *Let S be a subset of X containing a UP-ideal B of X. Then S is a strongly UP-ideal of X with respect to B if and only if S is a lower rough strongly UP-ideal of X.*

Proof. Let $x \in (0)_{\sim_B}$. Then $(x, 0) \in \sim_B$, that is, $x = 0 \cdot x \in B \subseteq S$. Thus $(0)_{\sim_B} \subseteq S$. Hence, $0 \in \sim_{B-}(S)$. Next, let $(z \cdot y) \cdot (z \cdot x) \in \sim_{B-}(S)$ and $y \in \sim_{B-}(S)$. Then $((z \cdot y) \cdot (z \cdot x))_{\sim_B} \subseteq S$ and $(y)_{\sim_B} \subseteq S$. Thus $y \in S$. We shall show that $x \in \sim_{B-}(S)$, that is, $(x)_{\sim_B} \subseteq S$. Let $a \in (x)_{\sim_B}$. Since $y \in (y)_{\sim_B}$ and $z \in (z)_{\sim_B}$, we have

$$\begin{aligned}
(z \cdot y) \cdot (z \cdot a) &\in [(z)_{\sim_B} \cdot (y)_{\sim_B}] \cdot [(z)_{\sim_B} \cdot (x)_{\sim_B}] \\
&\subseteq (z \cdot y)_{\sim_B} \cdot (z \cdot x)_{\sim_B} && \text{(By Lemma 3.6)} \\
&\subseteq ((z \cdot y) \cdot (z \cdot x))_{\sim_B} && \text{(By Lemma 3.6)} \\
&\subseteq S.
\end{aligned}$$

Thus $(z \cdot y) \cdot (z \cdot a) \in S$. Since S is a strongly UP-ideal of X, we have $a \in S$. Thus $(x)_{\sim_B} \subseteq S$. that is, $x \in \sim_{B-}(S)$. Hence, $\sim_{B-}(S)$ is a strongly UP-ideal of X. On the other hand, let S be a lower rough strongly UP-ideal of X. Then $\sim_{B-}(S)$ is a strongly UP-ideal of X. Thus $X = \sim_{B-}(S) \subseteq S \subseteq X$. Hence, $S = X$, it follows from Theorem 2.7 (3) that S is a strongly UP-ideal of X. $\square$

Theorem 4.26. *Let S be a strongly UP-ideal of X containing a UP-ideal B of X. Then $\sim_{B+}(S)$ is a strongly UP-ideal of X, that is, S is an upper rough strongly UP-ideal of X with respect to B.*

Proof. Since $0 \in (0)_{\sim_B}$ and $0 \in S$, we have $0 \in (0)_{\sim_B} \cap S \neq \emptyset$. Hence, $0 \in \sim_{B+}(S)$. Next, let $(z \cdot y) \cdot (z \cdot x) \in \sim_{B+}(S)$ and $y \in \sim_{B+}(S)$. Then $((z \cdot y) \cdot (z \cdot x))_{\sim_B} \cap S \neq \emptyset$ and $(y)_{\sim_B} \cap S \neq \emptyset$. We shall show that $x \in \sim_{B+}(S)$, that is, $(x)_{\sim_B} \cap S \neq \emptyset$. Since $((z \cdot y) \cdot (z \cdot x))_{\sim_B} \cap S \neq \emptyset$ and $(y)_{\sim_B} \cap S \neq \emptyset$ so we have $s_1, s_2 \in S$ such that $s_1 \in ((z \cdot y) \cdot (z \cdot x))_{\sim_B}$ and $s_2 \in (y)_{\sim_B}$. Thus $(s_1, (z \cdot y) \cdot (z \cdot x)) \in \sim_B$ and $(s_2, y) \in \sim_B$. Then $((z \cdot s_2) \cdot (z \cdot x), (z \cdot y) \cdot (z \cdot x)) \in \sim_B$. By transitive, we have $(s_1, (z \cdot s_2) \cdot (z \cdot x)) \in \sim_B$. Thus $(s_1)_{\sim_B} = ((z \cdot s_2) \cdot (z \cdot x))_{\sim_B}$. Since S is a strongly UP-ideal of X, we have S is a UP-filter of X. By Lemma 4.11, we have $(s_1)_{\sim_B} \subseteq S$. Thus $((z \cdot s_2) \cdot (z \cdot x))_{\sim_B} \subseteq S$. Since $(z \cdot s_2) \cdot (z \cdot x) \in ((z \cdot s_2) \cdot (z \cdot x))_{\sim_B} \subseteq S$ and S is a strongly UP-ideal of X, we have $x \in S$. Thus $x \in (x)_{\sim_B} \cap S \neq \emptyset$. Hence, $x \in \sim_{B+}(S)$. Therefore, $\sim_{B+}(S)$ is a strongly UP-ideal of X. □

Example 4.27. From Example 4.18, we have $S = \{0, 1, 2, 3, 5, 7\}$ is not a strongly UP-ideal of X but $\sim_{B+}(S) = \{0, 1, 2, 3, 4, 5, 6, 7\} = X$, it follows from Theorem 2.7 (3) that $\sim_{B+}(S)$ is a strongly UP-ideal of X, that is, S is an upper rough strongly UP-ideal of X with respect to B.

By Theorem 4.19, 4.21, 4.23 and 4.25 and Example 4.20, 4.22, and 4.24, we have that the notion of upper rough UP-subalgebras is a generalization of UP-subalgebras and rough UP-subalgebras, rough UP-filters is a generalization of UP-filters, rough UP-ideals is a generalization of UP-ideals, and rough strongly UP-ideals and strongly UP-ideals coincide. Hence, we have the following relation:

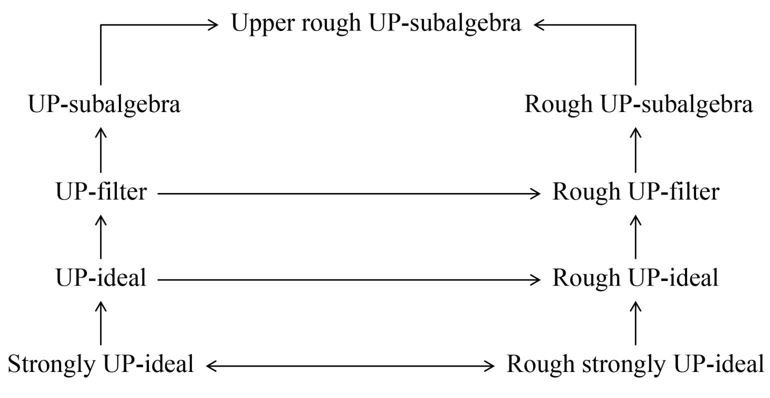


Figure 1: Relation 2

CHAPTER 5
Conclusions

From the study, we get the main results as the following:

1. Let A and B be nonempty subsets of a UP-algebra X. If ρ is an equivalence relation on X, then the following statements hold:

   (1) $\rho_{-}(A) \subseteq A \subseteq \rho_{+}(A)$,

   (2) $A \subseteq B$ implies $\rho_{-}(A) \subseteq \rho_{-}(B)$ and $\rho_{+}(A) \subseteq \rho_{+}(B)$,

   (3) $\rho_{-}(A \cap B) = \rho_{-}(A) \cap \rho_{-}(B)$,

   (4) $\rho_{-}(A \cup B) \supseteq \rho_{-}(A) \cup \rho_{-}(B)$,

   (5) $\rho_{+}(A \cap B) \subseteq \rho_{+}(A) \cap \rho_{+}(B)$,

   (6) $\rho_{+}(A \cup B) = \rho_{+}(A) \cup \rho_{+}(B)$,

   (7) $\rho_{-}(A') \subseteq (\rho_{-}(A))'$,

   (8) $(\rho_{+}(A))' \subseteq \rho_{+}(A')$, and

   (9) $\rho_{-}(A - B) \subseteq \rho_{-}(A) - \rho_{-}(B)$.

2. If ρ is a congruence relation on X, then $(x)_\rho \cdot (y)_\rho \subseteq (x \cdot y)_\rho$ for all $x, y \in X$.

3. Let A and B be nonempty subsets of X. If ρ is a congruence relation on X, then $\rho_{+}(A) \cdot \rho_{+}(B) \subseteq \rho_{+}(A \cdot B)$.

4. Let ρ be a congruence relation on X. If C is a strongly UP-ideal of X, then C is a rough strongly UP-ideal of X.

5. Every rough strongly UP-ideal of X is a rough UP-ideal.

6. Every rough UP-ideal of X is a rough UP-filter.

7. Every rough UP-filter of X is a rough UP-subalgebra.

8. Let ρ be a congruence relation on X. If S is a UP-filter of X such that $(0)_\rho \subseteq S$, then $(s)_\rho \subseteq S$ for all $s \in S$.

9. If B and C are UP-ideals of X such that $B \subseteq C$, then $\sim_B \subseteq \sim_C$.

10. Every nonempty subset of X is definable with respect to $\{0\}$.

11. Let S be a UP-subalgebra of X containing a UP-ideal B of X. Then $\sim_{B+}(S)$ is a UP-subalgebra of X, that is, S is an upper rough UP-subalgebra of X with respect to B.

12. Let S be a UP-filter of X containing a UP-ideal B of X. Then

    (1) $\sim_{B-}(S)$ is a UP-filter of X,

    (2) $\sim_{B+}(S)$ is a UP-filter of X.

    Moreover, S is a rough UP-filter of X with respect to B.

13. Let S be a UP-ideal of X containing a UP-ideal B of X. Then

    (1) $\sim_{B-}(S)$ is a UP-ideal of X,

    (2) $\sim_{B+}(S)$ is a UP-ideal of X.

    Moreover, S is a rough UP-ideal of X with respect to B.

14. Let S be a subset of X containing a UP-ideal B of X. Then S is a strongly UP-ideal of X with respect to B if and only if S is a lower rough strongly UP-ideal of X.

15. Let S be a strongly UP-ideal of X containing a UP-ideal B of X. Then $\sim_{B+}(S)$ is a strongly UP-ideal of X, that is, S is an upper rough strongly UP-ideal of X with respect to B.

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] R. Biswas and S. Nanda, Rough groups and rough subgroups, Bull. Polish Acad. Sci. Math. **42** (1994), no. 3, 251–254.

[2] W. A. Dudek, Y. B. Jun, and H. S. Kim, Rough set theory applied to BCI-algebras, Quasigroups Relat. Syst. **9** (2002), 45–54.

[3] T. Guntasow, S. Sajak, A. Jomkham, and A. Iampan, Fuzzy translations of a fuzzy set in UP-algebras, J. Indones. Math. Soc. **23** (2017), no. 2, 1–19.

[4] Q. P. Hu and X. Li, On BCH-algebras, Math. Semin. Notes, Kobe Univ. **11** (1983), 313–320.

[5] A. Iampan, A new branch of the logical algebra: UP-algebras, J. Algebra Relat. Top. **5** (2017), no. 1, 35–54.

[6] Y. Imai and K. Iséki, On axiom system of propositional calculi, XIV, Proc. Japan Acad. **42** (1966), no. 1, 19–22.

[7] K. Iséki, An algebra related with a propositional calculus, Proc. Japan Acad. **42** (1966), no. 1, 26–29.

[8] Y. B. Jun, Roughness of ideals in BCK-algebras, Sci. Math. Japon. **7** (2002), 115–119.

[9] S. Keawrahun and U. Leerawat, On isomorphisms of SU-algebras, Sci. Magna **7** (2011), no. 2, 39–44.

[10] N. Kuroki, Rough ideals in semigroups, Inform. Sci. **100** (1997), 139–163.

[11] N. Kuroki and J. N. Mordeson, Structure of rough sets and rough groups, J. Fuzzy Math. **5** (1997), 183–191.

[12] Z. Pawlak, Rough sets, Int. J. Inform. Comp. Sci. **11** (1982), 341–356.

[13] C. Prabpayak and U. Leerawat, On ideals and congruences in KU-algebras, Sci. Magna **5** (2009), no. 1, 54–57.

[14] J. Somjanta, N. Thuekaew, P. Kumpeangkeaw, and A. Iampan, Fuzzy sets in UP-algebras, Ann. Fuzzy Math. Inform. **12** (2016), no. 6, 739–756.

APPENDIX

Writing for Publication: Rough Set Theory applied to UP-Algebras*

Theeyarat Klinseesook (theeyarat.kss@gmail.com)
Sukhontha Bukok (sukhonthaswitch@gmail.com)
Aiyared Iampan†(aiyared.ia@up.ac.th)

Department of Mathematics, School of Science
University of Phayao, Phayao 56000, Thailand

Friday 27th April, 2018

Abstract

In this paper, rough set theory is applied to UP-algebras, proved some results and discussed the generalization of some notions of rough UP-subalgebras, rough UP-filters, rough UP-ideals and rough strongly UP-ideals. Furthermore, we discuss the relation between rough UP-subalgebras (resp., rough UP-filters, rough UP-ideals and rough strongly UP-ideals) and UP-subalgebras (resp., UP-filters, UP-ideals and strongly UP-ideals) and present some examples.

Mathematics Subject Classification: 03G25
Keywords: UP-algebra, rough UP-subalgebra, rough UP-filter, rough UP-ideal, rough strongly UP-ideal

---

1 Introduction

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [6], BCI-algebras [7], BCH-algebras [4], KU-algebras [13], SU-algebras [9], UP-algebras [5] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [7] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [6, 7] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

The notion of rough sets was first considered by Pawlak [12] in 1982. After the introduction of the notion of rough sets, several authors were conducted on the generalizations of the notion of rough sets and application to many many algebraic structures such as: In 1994,

*This work was financially supported by the University of Phayao.
†Corresponding author.

Biswas and Nanda [1] introduced and discussed the notion of rough groups and rough subgroups. Rough set theory is applied to semigroups and groups by Kuroki [10], and Kuroki and Mordeson [11] in 1997. In 2002, Jun [8] and Dudek et al. [2] applied rough set theory to BCK-algebras and BCI-algebras. In 2016, Mao and Zhou [8] applied rough set theory to pseudo-BCK-algebras.

In this paper, we apply the rough set theory to UP-algebras, introduce the notion of upper and lower rough UP-subalgebras (resp., rough UP-filters, rough UP-ideals and rough strongly UP-ideals) of UP-algebras, and discuss some of their important properties and its generalizations.

2 Basic Results on UP-Algebras

An algebra $X = (X, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* [5] where X is a nonempty set, $\cdot$ is a binary operation on X, and 0 is a fixed element of X (i.e., a nullary operation) if it satisfies the following axioms: for any $x, y, z \in X$,

(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,

(UP-2) $0 \cdot x = x$,

(UP-3) $x \cdot 0 = 0$, and

(UP-4) $x \cdot y = 0$ and $y \cdot x = 0$ imply $x = y$.

From [5], we know that the notion of UP-algebras is a generalization of KU-algebras.

Example 2.1. [5] Let X be a universal set. Define two binary operations $\cdot$ and $*$ on the power set of X by putting $A \cdot B = B \cap A'$ and $A * B = B \cup A'$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X), \cdot, \emptyset)$ and $(\mathcal{P}(X), *, X)$ are UP-algebras and we shall call it the *power UP-algebra of type 1* and the *power UP-algebra of type 2*, respectively.

The following is an important property of UP-algebras.

Proposition 2.2. *[5] In a UP-algebra X, the following properties hold: for any $x, y, z \in X$,*

(1) $x \cdot x = 0$,

(2) $x \cdot y = 0$ *and* $y \cdot z = 0$ *imply* $x \cdot z = 0$,

(3) $x \cdot y = 0$ *implies* $(z \cdot x) \cdot (z \cdot y) = 0$,

(4) $x \cdot y = 0$ *implies* $(y \cdot z) \cdot (x \cdot z) = 0$,

(5) $x \cdot (y \cdot x) = 0$,

(6) $(y \cdot x) \cdot x = 0$ *if and only if* $x = y \cdot x$, *and*

(7) $x \cdot (y \cdot y) = 0$.

In what follows, let X denote a UP-algebra unless otherwise specified.

Definition 2.3. [5] A subset S of X is called a *UP-subalgebra* of X if the constant 0 of X is in S, and $(S, \cdot, 0)$ itself forms a UP-algebra.

Iampan [5] proved the useful criteria that a nonempty subset S of a UP-algebra $X = (X, \cdot, 0)$ is a UP-subalgebra of X if and only if S is closed under the $\cdot$ multiplication on X.

Definition 2.4. [14] A subset F of X is called a *UP-filter* of X if it satisfies the following properties:

(1) the constant 0 of X is in F, and

(2) for any $x, y \in X, x \cdot y \in F$ and $x \in F$ imply $y \in F$.

Definition 2.5. [5] A subset B of X is called a *UP-ideal* of X if it satisfies the following properties:

(1) the constant 0 of X is in B, and

(2) for any $x, y, z \in X, x \cdot (y \cdot z) \in B$ and $y \in B$ imply $x \cdot z \in B$.

Definition 2.6. [3] A subset C of X is called a *strongly UP-ideal* of X if it satisfies the following properties:

(1) the constant 0 of X is in C, and

(2) for any $x, y, z \in X, (z \cdot y) \cdot (z \cdot x) \in C$ and $y \in C$ imply $x \in C$.

Theorem 2.7. *[3] The following statements hold:*

(1) every UP-filter of X is a UP-subalgebra,

(2) every UP-ideal of X is a UP-filter, and

(3) every strongly UP-ideal of X is a UP-ideal. Moreover, a UP-algebra X is the only one strongly UP-ideal of itself.

3 Rough UP-Algebras

Definition 3.1. Let X be a set and ρ an equivalence relation on X and let $\mathcal{P}(X)$ denote the power set of X. If $x \in X$, then the ρ-class of x is the set $(x)_\rho$ defined as follows:

$$(x)_\rho = \{y \in X \mid (x, y) \in \rho\}.$$

Define the functions ρ_- and ρ_+ from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ putting for every $S \in \mathcal{P}(X)$,

$$\rho_-(S) = \{x \in X \mid (x)_\rho \subseteq S\},$$
$$\rho_+(S) = \{x \in X \mid (x)_\rho \cap S \neq \emptyset\}.$$

$\rho_-(S)$ is called the *lower approximation of* S while $\rho_+(S)$ is called the *upper approximation of* S. The set S is called *definable* if $\rho_-(S) = \rho_+(S)$ and *rough* otherwise. The pair (X, ρ) is called an *approximation space*.

Proposition 3.2. *Let A and B be nonempty subsets of a UP-algebra X. If ρ is an equivalence relation on X, then the following statements hold:*

(1) $\rho_-(A) \subseteq A \subseteq \rho_+(A)$,

(2) $A \subseteq B$ implies $\rho_-(A) \subseteq \rho_-(B)$ and $\rho_+(A) \subseteq \rho_+(B)$,

(3) $\rho_-(A \cap B) = \rho_-(A) \cap \rho_-(B)$,

(4) $\rho_-(A \cup B) \supseteq \rho_-(A) \cup \rho_-(B)$,

(5) $\rho_+(A \cap B) \subseteq \rho_+(A) \cap \rho_+(B)$,

(6) $\rho_+(A \cup B) = \rho_+(A) \cup \rho_+(B)$,

(7) $\rho_-(A') \subseteq (\rho_-(A))'$,

(8) $(\rho_+(A))' \subseteq \rho_+(A')$, *and*

(9) $\rho_-(A - B) \subseteq \rho_-(A) - \rho_-(B)$.

Proof. (1) Let $x \in \rho_-(A)$. Then $(x)_\rho \subseteq A$. By reflexivity, $(x, x) \in \rho$ so $x \in (x)_\rho$. Thus $x \in A$, that is, $\rho_-(A) \subseteq A$. Let $y \in A$. By reflexivity, $(y, y) \in \rho$ so $y \in (y)_\rho$. Thus $y \in (y)_\rho \cap A \neq \emptyset$. So $y \in \rho_+(A)$, that is, $A \subseteq \rho_+(A)$. Therefore, $\rho_-(A) \subseteq A \subseteq \rho_+(A)$.

(2) Assume that $A \subseteq B$. Let $x \in \rho_-(A)$. Then $(x)_\rho \subseteq A \subseteq B$. Thus $x \in \rho_-(B)$, that is, $\rho_-(A) \subseteq \rho_-(B)$. Let $x \in \rho_+(A)$. Then $(x)_\rho \cap A \neq \emptyset$, so there is $y \in (x)_\rho \cap A$. Thus $y \in (x)_\rho$ and $y \in A \subseteq B$, that is, $y \in (x)_\rho \cap B \neq \emptyset$. Thus $x \in \rho_+(B)$. Hence, $\rho_+(A) \subseteq \rho_+(B)$.

(3) By Proposition 3.2 (2), we get $\rho_-(A \cap B) \subseteq \rho_-(A)$ and $\rho_-(A \cap B) \subseteq \rho_-(B)$. Hence, $\rho_-(A \cap B) \subseteq \rho_-(A) \cap \rho_-(B)$. On the other hand, let $x \in \rho_-(A) \cap \rho_-(B)$. Then $x \in \rho_-(A)$ and $x \in \rho_-(B)$. Thus $(x)_\rho \subseteq A$ and $(x)_\rho \subseteq B$. So $(x)_\rho \subseteq A \cap B$, that is, $x \in \rho_-(A \cap B)$. Therefore, $\rho_-(A) \cap \rho_-(B) \subseteq \rho_-(A \cap B)$. Hence, $\rho_-(A) \cap \rho_-(B) = \rho_-(A \cap B)$.

(4) By Proposition 3.2 (2), we get $\rho_-(A) \subseteq \rho_-(A \cup B)$ and $\rho_-(B) \subseteq \rho_-(A \cup B)$. Hence, $\rho_-(A) \cup \rho_-(B) \subseteq \rho_-(A \cup B)$.

(5) By Proposition 3.2 (2), we get $\rho_+(A \cap B) \subseteq \rho_+(A)$ and $\rho_+(A \cap B) \subseteq \rho_+(B)$. Hence, $\rho_+(A \cap B) \subseteq \rho_+(A) \cap \rho_+(B)$.

(6) Let $x \in \rho_+(A \cup B)$. Then $(x)_\rho \cap (A \cup B) \neq \emptyset$. Thus $((x)_\rho \cap A) \cup ((x)_\rho \cap B) \neq \emptyset$, we have $(x)_\rho \cap A \neq \emptyset$ or $(x)_\rho \cap B \neq \emptyset$. Hence, $x \in \rho_+(A)$ or $x \in \rho_+(B)$. Therefore, $x \in \rho_+(A) \cup \rho_+(B)$, that is, $\rho_+(A \cup B) \subseteq \rho_+(A) \cup \rho_+(B)$. On the other hand, $\rho_+(A) \subseteq \rho_+(A \cup B)$ and $\rho_+(B) \subseteq \rho_+(A \cup B)$ by Proposition 3.2 (2). Hence, $\rho_+(A) \cup \rho_+(B) \subseteq \rho_+(A \cup B)$, that is, $\rho_+(A \cup B) = \rho_+(A) \cup \rho_+(B)$.

(7) Let $x \in \rho_-(A')$. Then $(x)_\rho \subseteq A'$ and so $(x)_\rho \nsubseteq A$. Thus $x \notin \rho_-(A)$, that is, $x \in (\rho_-(A))'$. Hence, $\rho_-(A') \subseteq (\rho_-(A))'$.

(8) Let $x \in (\rho_+(A))'$. Then $x \notin \rho_+(A)$ and so $(x)_\rho \cap A = \emptyset$. Thus $x \notin A$, that is, $x \in A'$. Therefore, $(x)_\rho \cap A' \neq \emptyset$, that is, $x \in \rho_+(A')$. Hence, $(\rho_+(A))' \subseteq \rho_+(A')$.

(9) Now,

$$\begin{aligned}
\rho_-(A - B) &= \rho_-(A \cap B') \\
&= \rho_-(A) \cap \rho_-(B') && ((3)) \\
&\subseteq \rho_-(A) \cap (\rho_-(B))' && ((7)) \\
&= \rho_-(A) - \rho_-(B).
\end{aligned}$$

Hence, $\rho_-(A - B) \subseteq \rho_-(A) - \rho_-(B)$. □

Remark 3.3. Let ρ be an equivalence relation on a set X. Then $\rho_-(X) = X = \rho_+(X)$.

Proof. By Proposition 3.2 (1), we have $\rho_-(X) \subseteq X \subseteq \rho_+(X)$ and $\rho_+(X) \subseteq X$. Thus $X = \rho_+(X)$. We shall show that $X \subseteq \rho_-(X)$. Let $x \in X$. Then $(x)_\rho \subseteq X$. Thus $x \in \rho_-(X)$, that is, $X \subseteq \rho_-(X)$. Hence, $\rho_-(X) = X = \rho_+(X)$. □

Definition 3.4. Let ρ be a congruence relation on X. Then the set of all ρ-classes is called the *quotient set* of X by ρ, and is denoted by X/ρ. That is,

$$X/\rho = \{(x)_\rho \mid x \in X\}.$$

Define a binary operation $*$ on X/ρ by $(x)_\rho * (y)_\rho = (x \cdot y)_\rho$ for all $x, y \in X$. Then $(X/\rho, *, (0)_\rho)$ is an algebra of type $(2, 0)$. Indeed, let $(x_1)_\rho = (x_2)_\rho$ and $(y_1)_\rho = (y_2)_\rho$. Then $(x_1, x_2) \in \rho$ and $(y_1, y_2) \in \rho$, so $(x_1 \cdot y_1, x_2 \cdot y_2) \in \rho$ because ρ is a congruence relation on X. Hence, $(x_1)_\rho * (y_1)_\rho = (x_1 \cdot y_1)_\rho = (x_2 \cdot y_2)_\rho = (x_2)_\rho * (y_2)_\rho$.

Definition 3.5. For nonempty subsets A and B of a UP-algebra $X = (X, \cdot, 0)$, we denote

$$A \cdot B = \{a \cdot b \mid a \in A \text{ and } b \in B\}.$$

Lemma 3.6. *If ρ is a congruence relation on X, then $(x)_\rho \cdot (y)_\rho \subseteq (x \cdot y)_\rho$ for all $x, y \in X$.*

Proof. Let $x, y \in X$ and $t \in (x)_\rho \cdot (y)_\rho$. Then $t = a \cdot b$ for some $a \in (x)_\rho$ and $b \in (y)_\rho$. Thus $(a, x) \in \rho$ and $(b, y) \in \rho$. So $(a \cdot b, x \cdot y) \in \rho$, that is, $t = a \cdot b \in (x \cdot y)_\rho$. Therefore, $(x)_\rho \cdot (y)_\rho \subseteq (x \cdot y)_\rho$. □

Example 3.7. Let $X = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

Then $(X, \cdot, 0)$ is a UP-algebra. We see that

$$\rho = \{(0, 0), (1, 1), (2, 2), (3, 3), (0, 1), (1, 0)\}$$

is a congruence relation on X. Thus

$$(0)_\rho = \{0, 1\}, (1)_\rho = \{0, 1\}, (2)_\rho = \{2\}, \text{ and } (3)_\rho = \{3\}.$$

Since $(2 \cdot 2)_\rho = (0)_\rho = \{0, 1\}$ and $(2)_\rho \cdot (2)_\rho = \{2\} \cdot \{2\} = \{0\}$, we have $(2)_\rho \cdot (2)_\rho = \{0\} \not\supseteq \{0, 1\} = (2 \cdot 2)_\rho$.

Proposition 3.8. *Let A and B be nonempty subsets of X. If ρ is a congruence relation on X, then $\rho_+(A) \cdot \rho_+(B) \subseteq \rho_+(A \cdot B)$.*

Proof. Let $t \in \rho_+(A) \cdot \rho_+(B)$. Then $t = x \cdot y$ for some $x \in \rho_+(A)$ and $y \in \rho_+(B)$. Thus $(x)_\rho \cap A \neq \emptyset$ and $(y)_\rho \cap B \neq \emptyset$, that is, $a \in (x)_\rho \cap A$ and $b \in (y)_\rho \cap B$ for some $a, b \in X$. By Lemma 3.6, we have $a \cdot b \in (x)_\rho \cdot (y)_\rho \subseteq (x \cdot y)_\rho$ and $a \cdot b \in A \cdot B$, so $a \cdot b \in (x \cdot y)_\rho \cap (A \cdot B) \neq \emptyset$. Thus $(t)_\rho \cap (A \cdot B) = (x \cdot y)_\rho \cap (A \cdot B) \neq \emptyset$, that is, $t \in \rho_+(A \cdot B)$. Hence, $\rho_+(A) \cdot \rho_+(B) \subseteq \rho_+(A \cdot B)$. □

Example 3.9. From Example 3.7, let $A = \{3\}$ and $B = \{2, 3\}$. Then $A \cdot B = \{0, 2\}$, $\rho_+(A) = \{3\}$ and $\rho_+(B) = \{2, 3\}$. Thus $\rho_+(A) \cdot \rho_+(B) = \{0, 2\} \not\supseteq \{0, 1, 2\} = \rho_+(A \cdot B)$.

4 Main Results

In the next part, we will research and analysis upper and lower rough UP-subalgebras (resp., rough UP-filters, rough UP-ideals and rough strongly UP-ideals) of UP-algebras, and discuss some of their important properties and its generalizations.

Definition 4.1. Let S be a nonempty subset of X and ρ an equivalence relation on X. Then S is called

(1) an *upper rough UP-subalgebra* of X if $\rho_+(S)$ is a UP-subalgebra of X,

(2) an *upper rough UP-filter* of X if $\rho_+(S)$ is a UP-filter of X,

(3) an *upper rough UP-ideal* of X if $\rho_+(S)$ is a UP-ideal of X,

(4) an *upper rough strongly UP-ideal* of X if $\rho_+(S)$ is a strongly UP-ideal of X,

(5) a *lower rough UP-subalgebra* of X if $\rho_-(S)$ is a UP-subalgebra of X when $\rho_-(S)$ is nonempty,

(6) a *lower rough UP-filter* of X if $\rho_-(S)$ is a UP-filter of X when $\rho_-(S)$ is nonempty,

(7) a *lower rough UP-ideal* of X if $\rho_-(S)$ is a UP-ideal of X when $\rho_-(S)$ is nonempty,

(8) a *lower rough strongly UP-ideal* of X if $\rho_-(S)$ is a strongly UP-ideal of X when $\rho_-(S)$ is nonempty,

(9) a *rough UP-subalgebra* of X if it is both an upper and a lower rough UP-subalgebra of X,

(10) a *rough UP-filter* of X if it is both an upper and a lower rough UP-filter of X,

(11) a *rough UP-ideal* of X if it is both an upper and a lower rough UP-ideal of X, and

(12) a *rough strongly UP-ideal* of X if it is both an upper and a lower rough strongly UP-ideal of X.

Example 4.2. Let $X = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	2
2	0	1	0	3	1
3	0	1	2	0	4
4	0	0	0	3	0

Then $(X, \cdot, 0)$ is a UP-algebra. We see that

$$\rho = \{(0,0), (1,1), (2,2), (3,3), (4,4), (0,2), (2,0), (1,4), (4,1)\}$$

is a congruence relation on X. Thus

$$(0)_\rho = (2)_\rho = \{0, 2\}, (3)_\rho = \{3\}, \text{ and } (1)_\rho = (4)_\rho = \{1, 4\}.$$

We have

(1) $S := \{0, 3\}$ is a UP-ideal (resp., UP-filter and UP-subalgebra) of X but $\rho_-(S) = \{3\}$ is not a UP-ideal (resp., UP-filter and UP-subalgebra) of X. Thus S is not a lower rough UP-ideal (resp., lower rough UP-filter and lower rough UP-subalgebra) of X. Hence, S is not a rough UP-ideal (resp., rough UP-filter and rough UP-subalgebra) of X.

(2) $S := \{0, 2, 4\}$ is not a UP-subalgebra (resp., UP-filter and UP-ideal) of X but $\rho_-(S) = \{0, 2\}$ is a UP-subalgebra (resp., UP-filter and UP-ideal) and $\rho_+(S) = \{0, 1, 2, 4\}$ is a UP-subalgebra (resp., UP-filter and UP-ideal) of X. Thus S is both a lower and an upper rough UP-subalgebra (resp., rough UP-filter and rough UP-ideal) of X. Hence, S is a rough UP-subalgebra (resp., rough UP-filter and rough UP-ideal) of X.

(3) $S := \{0, 1\}$ is a UP-ideal (resp., UP-filter and UP-subalgebra) of X. Then $\rho_-(S) = \emptyset$ and $\rho_+(S) = \{0, 1, 2, 4\}$. Thus S is both a lower and an upper rough UP-ideal (resp., rough UP-filter and rough UP-subalgebra) of X. Hence, S is a rough UP-ideal (resp., rough UP-filter and rough UP-subalgebra) of X.

(4) If $\rho = X \times X$, then $(0)_\rho = (1)_\rho = (2)_\rho = (3)_\rho = X$. Thus $S := \{1, 3\}$ is not a UP-ideal (resp., UP-filter and UP-subalgebra) of X, and $\rho_-(S) = \emptyset$ and $\rho_+(S) = X$, that is, S is both a lower and an upper rough UP-ideal of X. Hence, S is a rough UP-ideal (resp., rough UP-filter and rough UP-subalgebra) of X.

Theorem 4.3. *Let ρ be a congruence relation on X. If C is a strongly UP-ideal of X, then C is a rough strongly UP-ideal of X.*

Proof. Assume that C is a strongly UP-ideal of X. By Theorem 2.7 (3), we have $C = X$. By Remark 3.3, we have $\rho_-(C) = X = \rho_+(C)$. By Theorem 2.7 (3) again, we have $\rho_-(C)$ and $\rho_+(C)$ are strongly UP-ideals of X. Therefore, C is a rough strongly UP-ideal of X. □

Example 4.4. From Example 4.2 (4), we have $C := \{0, 1, 2\}$ is not a strongly UP-ideal of X. Since $\rho_-(C) = \emptyset$ and $\rho_+(C) = X$, we have C is both a lower and an upper rough strongly UP-ideal of X. Hence, C is a rough strongly UP-ideal of X.

Theorem 4.5. *Every rough strongly UP-ideal of X is a rough UP-ideal.*

Proof. Let S be a rough strongly UP-ideal of X. Then $\rho_-(S)$ (if $\rho_-(S)$ is nonempty) and $\rho_+(S)$ are strongly UP-ideals of X. By Theorem 2.7 (3), $\rho_-(S)$ (if $\rho_-(S)$ is nonempty) and $\rho_+(S)$ are UP-ideals of X. Hence, S is a rough UP-ideal of X. □

Example 4.6. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	0	0	2	3	2	5
2	0	1	0	3	1	5
3	0	1	2	0	4	5
4	0	0	0	3	0	5
5	0	0	2	0	2	0

Then $(X, \cdot, 0)$ is a UP-algebra. We see that

$$\rho = \{(0,0), (1,1), (2,2), (3,3), (4,4), (5,5), (0,2), (2,0), (1,4), (4,1)\}$$

is a congruence relation on X. Thus

$$(0)_\rho = (2)_\rho = \{0, 2\},\ (1)_\rho = (4)_\rho = \{1, 4\},\ (3)_\rho = \{3\}, \text{ and } (5)_\rho = \{5\}.$$

If $S = \{0, 2, 4\}$, then $\rho_-(S) = \{0, 2\}$ and $\rho_+(S) = \{0, 1, 2, 4\}$. Thus $\rho_-(S)$ and $\rho_+(S)$ are UP-ideals of X. Hence, S is a rough UP-ideal of X. Since $\rho_-(S) \neq X$ and $\rho_+(S) \neq X$, it follows from Theorem 2.7 (3) that $\rho_-(S)$ and $\rho_+(S)$ are not a strongly UP-ideal of X. Hence, S is a rough UP-ideal of X but is not a rough strongly UP-ideal.

Theorem 4.7. *Every rough UP-ideal of X is a rough UP-filter.*

Proof. Let S be a rough UP-ideal of X. Then $\rho_-(S)$ (if $\rho_-(S)$ is nonempty) and $\rho_+(S)$ are UP-ideals of X. By Theorem 2.7 (2), we have $\rho_-(S)$ (if $\rho_-(S)$ is nonempty) and $\rho_+(S)$ are UP-filters of X. Hence, S is a rough UP-filter of X. □

Example 4.8. Let $X = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	2
3	0	1	0	0

Then $(X, \cdot, 0)$ is a UP-algebra. We see that

$$\rho = \{(0,0),(1,1),(2,2),(3,3)\}$$

is a congruence relation on X. Thus

$$(0)_\rho = \{0\}, (1)_\rho = \{1\}, (2)_\rho = \{2\}, \text{ and } (3)_\rho = \{3\}.$$

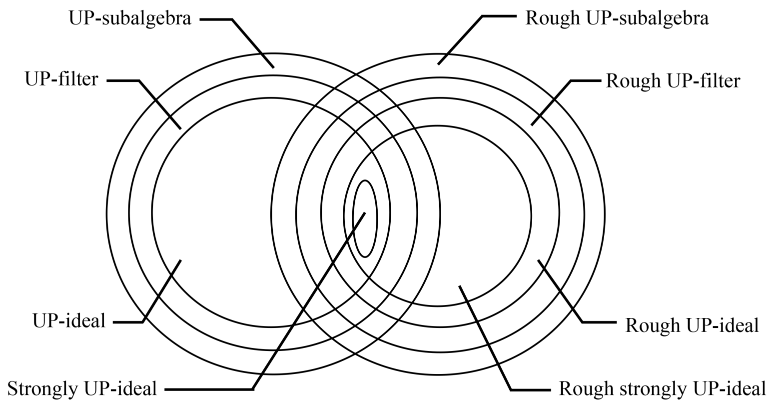
If $S = \{0,1\}$, then $\rho_-(S) = \{0,1\} = \rho_+(S)$. Thus $\rho_-(S)$ and $\rho_+(S)$ are UP-filters of X but are not a UP-ideal. Hence, S is a rough UP-filter of X but is not a rough UP-ideal.

Theorem 4.9. *Every rough UP-filter of X is a rough UP-subalgebra.*

Proof. Let S be a rough UP-filter of X. Then $\rho_-(S)$ (if $\rho_-(S)$ is nonempty) and $\rho_+(S)$ are UP-filters of X. By Theorem 2.7 (1), we have $\rho_-(S)$ (if $\rho_-(S)$ is nonempty) and $\rho_+(S)$ are UP-subalgebras of X. Hence, S is a rough UP-subalgebra of X. □

Example 4.10. From Example 4.6, if $S = \{0,1,2,5\}$, then $\rho_-(S) = \{0,2,5\}$ and $\rho_+(S) = \{0,1,2,4,5\}$. Thus $\rho_-(S)$ and $\rho_+(S)$ are UP-subalgebras of X but are not a UP-filter. Hence, S is a rough UP-subalgebra of X but is not a rough UP-filter.

By Theorem 4.5, 4.7, and 4.9 and Example 4.6, 4.8 and 4.10, we have that the notion of rough UP-subalgebras is a generalization of rough UP-filters, the notion of rough UP-filters is a generalization of rough UP-ideals, and the notion of rough UP-ideals is a generalization of rough strongly UP-ideals. By Example 4.2, the notions of UP-subalgebras (resp., UP-filters and UP-ideals) and rough UP-subalgebras (resp., rough UP-filters and rough UP-ideals) are not identical. Hence, we have the following relation:



Lemma 4.11. *Let ρ be a congruence relation on X. If S is a UP-filter of X such that $(0)_\rho \subseteq S$, then $(s)_\rho \subseteq S$ for all $s \in S$.*

Proof. Assume that $(s)_\rho \nsubseteq S$ for some $s \in S$. Then there is $x \in (s)_\rho$ but $x \notin S$, so $(x,s) \in \rho$. Since ρ is a congruence relation on X, we have $(s \cdot x, 0) = (s \cdot x, s \cdot s) \in \rho$. that is, $s \cdot x \in (s \cdot x)_\rho = (0)_\rho \subseteq S$. Since S is a UP-filter of X, we have $x \in S$ which is a contradiction. Hence $(s)_\rho \subseteq S$ for all $s \in S$. □

Definition 4.12. Let B be a UP-ideal of X. Define the binary relation $\sim_B$ on X as follows: for all $x, y \in X$,

$$x \sim_B y \text{ if and only if } x \cdot y \in B \text{ and } y \cdot x \in B. \tag{4.1}$$

$\sim_{B-}(S)$ is called the *lower approximation* of S by B while $\sim_{B+}(S)$ is called the *upper approximation* of S by B. The set S is called *definable* with respect to B if $\sim_{B-}(S) = \sim_{B+}(S)$ and *rough* with respect to B otherwise.

Iampan [5] proved that $\sim_B$ is a congruence relation on X.

Lemma 4.13. *If B and C are UP-ideals of X such that $B \subseteq C$, then $\sim_B \subseteq \sim_C$.*

Proof. Let $(x, y) \in \sim_B$. Then $x \cdot y, y \cdot x \in B \subseteq C$. Thus $(x, y) \in \sim_C$. Hence $\sim_B \subseteq \sim_C$. $\square$

Proposition 4.14. *Every nonempty subset of X is definable with respect to $\{0\}$.*

Proof. Let S be a nonempty subset of X. If $a \in (x)_{\sim_{\{0\}}}$, then $(a, x) \in \{0\}$. Thus $a \cdot x = 0 = x \cdot a$. By (UP-4), we have $a = x$. Thus $(x)_{\sim_{\{0\}}} = \{x\}$ for all $x \in X$. Now,

$$\begin{aligned} {\sim_{\{0\}}}_{-}(S) &= \{x \in X \mid (x)_{\sim_{\{0\}}} \subseteq S\} \\ &= \{x \in X \mid \{x\} \subseteq S\} \\ &= \{x \in X \mid x \in S\} \\ &= S \end{aligned}$$

and

$$\begin{aligned} {\sim_{\{0\}}}_{+}(S) &= \{x \in X \mid (x)_{\sim_{\{0\}}} \cap S \neq \emptyset\} \\ &= \{x \in X \mid \{x\} \cap S \neq \emptyset\} \\ &= \{x \in X \mid x \in S\} \\ &= S. \end{aligned}$$

Hence, ${\sim_{\{0\}}}_{-}(S) = S = {\sim_{\{0\}}}_{+}(S)$, that is, S is definable with respect to $\{0\}$. $\square$

Theorem 4.15. *[5] Let B be a UP-ideal of X. Then the following statements hold:*

(1) the $\sim_B$-class $(0)_{\sim_B}$ is a UP-ideal and a UP-subalgebra of X which $B = (0)_{\sim_B}$,

(2) a $\sim_B$-class $(x)_{\sim_B}$ is a UP-ideal of X if and only if $x \in B$,

(3) a $\sim_B$-class $(x)_{\sim_B}$ is a UP-subalgebra of X if and only if $x \in B$, and

(4) $(X/\sim_B, *, (0)_{\sim_B})$ is a UP-algebra under the $*$ multiplication defined by $(x)_{\sim_B} * (y)_{\sim_B} = (x \cdot y)_{\sim_B}$ for all $x, y \in X$, called the quotient UP-algebra of X induced by the congruence relation $\sim_B$.

Remark 4.16. If B is a UP-ideal of X, then ${\sim_B}_{-}(B) = B = {\sim_B}_{+}(B)$. that is, B is definable with respect to itself.

Proof. Assume that B is a UP-ideal of X. Let $x \in B$. Then $x \cdot 0 = 0 \in B$ and $0 \cdot x = x \in B$, that is, $(x, 0) \in \sim_B$ it follows from Theorem 4.15 that $B = (0)_{\sim_B} = (x)_{\sim_B}$, so $x \in {\sim_B}_{-}(B)$. Hence, $B \subseteq {\sim_B}_{-}(B) \subseteq {\sim_B}_{+}(B)$. By Proposition 3.2 (1), we have ${\sim_B}_{-}(B) \subseteq B$ so ${\sim_B}_{-}(B) = B$. Finally, we shall show that ${\sim_B}_{+}(B) \subseteq B$. Let $x \in {\sim_B}_{+}(B)$. Then $(x)_{\sim_B} \cap B \neq \emptyset$, so there is $a_x \in (x)_{\sim_B}$ and $a_x \in B = (0)_{\sim_B}$. Thus $(x)_{\sim_B} = (a_x)_{\sim_B} = (0)_{\sim_B} = B$, so $x \in B$. Thus ${\sim_B}_{+}(B) \subseteq B$. Hence, $B = {\sim_B}_{+}(B)$. $\square$

Remark 4.17. Let S be a nonempty subset of X contained in a UP-ideal B of X. Then ${\sim_B}_{+}(S) = B$ and ${\sim_B}_{-}(S) = \emptyset$.

Proof. Let $x \in {\sim_B}_{+}(S)$. Then $(x)_{\sim_B} \cap S \neq \emptyset$. Since $S \subset B$, we have $(x)_{\sim_B} \cap B \neq \emptyset$. By Remark 4.16, we have $x \in {\sim_B}_{+}(B) = B$. Thus ${\sim_B}_{+}(S) \subseteq B$. Next, we shall show that $B \subseteq {\sim_B}_{+}(S)$. Let $x \in B$. By Theorem 4.15 (1), we have $x \in B = (0)_{\sim_B}$. Then $(x)_{\sim_B} = (0)_{\sim_B} = B$, so $(x)_{\sim_B} \cap B \neq \emptyset$. Thus $x \in {\sim_B}_{+}(S)$. Hence, ${\sim_B}_{+}(S) = B$. Finally, we shall show that ${\sim_B}_{-}(S) = \emptyset$. Let ${\sim_B}_{-}(S) \neq \emptyset$. Then there are $x \in {\sim_B}_{-}(S)$. Thus $(x)_{\sim_B} \subseteq S \subset B$. By Theorem 4.15 (1), we have $x \in B = (0)_{\sim_B}$. Then $(x)_{\sim_B} = (0)_{\sim_B} = B$ which is a contradiction. Hence, ${\sim_B}_{-}(S) = \emptyset$. $\square$

By Remark 4.17, we can see that S is a rough UP-ideal (resp., rough UP-filter and rough UP-subalgebra) of X with respect to B.

Example 4.18. Let $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	0	0	2	3	2	3	6	7
2	0	1	0	3	1	5	3	7
3	0	1	2	0	4	1	2	7
4	0	0	0	3	0	3	3	7
5	0	0	2	0	2	0	2	7
6	0	1	0	0	1	1	0	7
7	0	0	0	0	0	0	0	0

Then $(X, \cdot, 0)$ is a UP-algebra. Let $B = \{0, 2\}$. Then B is a UP-ideal of X, so $\sim_B$ is a congruence relation on X. Thus $(0)_{\sim_B} = (2)_{\sim_B} = \{0, 2\}$, $(1)_{\sim_B} = (4)_{\sim_B} = \{1, 4\}$, $(3)_{\sim_B} = (6)_{\sim_B} = \{3, 6\}$, $(5)_{\sim_B} = \{5\}$, and $(7)_{\sim_B} = \{7\}$. Let $S = \{0, 1, 2, 3, 4, 5\}$. Then S is a UP-subalgebra of X but $\sim_{B-}(S) = \{0, 1, 2, 4, 5\}$ is not a UP-subalgebra of X. Thus S is not a lower rough UP-subalgebra of X. Hence, S is not a rough UP-subalgebra of X.

Theorem 4.19. *Let S be a UP-subalgebra of X containing a UP-ideal B of X. Then $\sim_{B+}(S)$ is a UP-subalgebra of X, that is, S is an upper rough UP-subalgebra of X with respect to B.*

Proof. By Proposition 3.2 (1), we have $S \subseteq \sim_{B+}(S) \neq \emptyset$. Let $x, y \in \sim_{B+}(S)$. Then $(x)_{\sim_B} \cap S \neq \emptyset$ and $(y)_{\sim_B} \cap S \neq \emptyset$. Thus there exist $a_x, a_y \in S$ such that $a_x \in (x)_{\sim_B}$ and $a_y \in (y)_{\sim_B}$. By Lemma 3.6, we have $a_x \cdot a_y \in (x)_{\sim_B} \cdot (y)_{\sim_B} \subseteq (x \cdot y)_{\sim_B}$. Since S is a UP-subalgebra of X, we have $a_x \cdot a_y \in S$. Thus $a_x \cdot a_y \in (x \cdot y)_{\sim_B} \cap S \neq \emptyset$. Hence, $x \cdot y \in \sim_{B+}(S)$, that is, $\sim_{B+}(S)$ is a UP-subalgebra of X. $\square$

Example 4.20. From Example 4.18, we have $S = \{0, 2, 4\}$ is not a UP-subalgebra of X but $\sim_{B+}(S) = \{0, 1, 2, 4\}$ is a UP-subalgebra of X, that is, S is an upper rough UP-subalgebra of X with respect to B.

Theorem 4.21. *Let S be a UP-filter of X containing a UP-ideal B of X. Then*

(1) $\sim_{B-}(S)$ is a UP-filter of X,

(2) $\sim_{B+}(S)$ is a UP-filter of X.

Moreover, S is a rough UP-filter of X with respect to B.

Proof. (1) Let $x \in (0)_{\sim_B}$. Then $(x, 0) \in \sim_B$, that is, $x = 0 \cdot x \in B \subseteq S$. Thus $(0)_{\sim_B} \subseteq S$. Hence $0 \in \sim_{B-}(S)$. Next, let $x \cdot y \in \sim_{B-}(S)$ and $x \in \sim_{B-}(S)$. Then $(x \cdot y)_{\sim_B} \subseteq S$ and $(x)_{\sim_B} \subseteq S$. Thus $x \in S$. We shall show that $y \in \sim_{B-}(S)$, that is, $(y)_{\sim_B} \subseteq S$. Let $a_y \in (y)_{\sim_B}$. Since $x \in (x)_{\sim_B}$, it follows from Lemma 3.6 that $x \cdot a_y \in (x)_{\sim_B} \cdot (y)_{\sim_B} \subseteq (x \cdot y)_{\sim_B} \subseteq S$. Thus $x \cdot a_y \in S$. Since S is a UP-filter of X, we have $a_y \in S$. Thus $(y)_{\sim_B} \subseteq S$, that is, $y \in \sim_{B-}(S)$. Hence, $\sim_{B-}(S)$ is a UP-filter of X.

(2) Since $0 \in (0)_{\sim_B}$ and $0 \in S$, we have $0 \in (0)_{\sim_B} \cap S \neq \emptyset$. Hence, $0 \in \sim_{B+}(S)$. Next, let $x \cdot y \in \sim_{B+}(S)$ and $x \in \sim_{B+}(S)$. Then $(x \cdot y)_{\sim_B} \cap S \neq \emptyset$ and $(x)_{\sim_B} \cap S \neq \emptyset$. We shall show that $y \in \sim_{B+}(S)$, that is, $(y)_{\sim_B} \cap S \neq \emptyset$. Let $u, v \in S$ be such that $u \in (x \cdot y)_{\sim_B}$ and $v \in (x)_{\sim_B}$. Thus $(u, x \cdot y) \in \sim_B$ and $(v, x) \in \sim_B$, so $u \cdot (x \cdot y) \in B \subseteq S$ and $v \cdot x \in B \subseteq S$. Since $u, v \in S$ and S is a UP-filter of X, we have $x \cdot y \in S$ and $x \in S$ and so $y \in S$. Since $y \in (y)_{\sim_B}$, we have $y \in (y)_{\sim_B} \cap S \neq \emptyset$. Thus $y \in \sim_{B+}(S)$. Hence, $\sim_{B+}(S)$ is a UP-filter of X. $\square$

Example 4.22. From Example 4.18, let $S = \{0, 2, 3\}$. Then S is not a UP-filter of X, But $\sim_{B-}(S) = \{0, 2\}$ and $\sim_{B+}(S) = \{0, 2, 3, 6\}$ are UP-filter of X, that is, S is both a lower and an upper rough UP-filter of X with respect to B. Hence, S is a rough UP-filter of X with respect to B.

Theorem 4.23. *Let S be a UP-ideal of X containing a UP-ideal B of X. Then*

(1) $\sim_{B-}(S)$ is a UP-ideal of X,

(2) $\sim_{B+}(S)$ is a UP-ideal of X.

Moreover, S is a rough UP-ideal of X with respect to B.

Proof. (1) Let $x \in (0)_{\sim_B}$. Then $(x, 0) \in \sim_B$, that is, $x = 0 \cdot x \in B \subseteq S$. Thus $(0)_{\sim_B} \subseteq S$. Hence, $0 \in \sim_{B-}(S)$. Next, let $x \cdot (y \cdot z) \in \sim_{B-}(S)$ and $y \in \sim_{B-}(S)$. Then $(x \cdot (y \cdot z))_{\sim_B} \subseteq S$ and $(y)_{\sim_B} \subseteq S$. Thus $y \in S$. We shall show that $x \cdot z \in \sim_{B-}(S)$, that is, $(x \cdot z)_{\sim_B} \subseteq S$. Since $x \cdot (y \cdot z) \in (x \cdot (y \cdot z))_{\sim_B} \subseteq S$ and S is a UP-ideal of X, we have $x \cdot z \in S$. By Lemma 4.11, we have $(x \cdot z)_{\sim_B} \subseteq S$. Thus $x \cdot z \in \sim_{B-}(S)$. Hence, $\sim_{B-}(S)$ is a UP-ideal of X.

(2) Since $0 \in (0)_{\sim_B}$ and $0 \in S$, we have $0 \in (0)_{\sim_B} \cap S \neq \emptyset$. Hence, $0 \in \sim_{B+}(S)$. Next, let $x \cdot (y \cdot z) \in \sim_{B+}(S)$ and $y \in \sim_{B+}(S)$. Then $(x \cdot (y \cdot z))_{\sim_B} \cap S \neq \emptyset$ and $(y)_{\sim_B} \cap S \neq \emptyset$. We shall show that $x \cdot z \in \sim_{B+}(S)$, that is, $(x \cdot z)_{\sim_B} \cap S \neq \emptyset$. Since $(x \cdot (y \cdot z))_{\sim_B} \cap S \neq \emptyset$ and $(y)_{\sim_B} \cap S \neq \emptyset$ so we have $s_1, s_2 \in S$ such that $s_1 \in (x \cdot (y \cdot z))_{\sim_B}$ and $s_2 \in (y)_{\sim_B}$. Thus $(s_1, x \cdot (y \cdot z)) \in \sim_B$ and $(s_2, y) \in \sim_B$. so $(x \cdot (s_2 \cdot z), x \cdot (y \cdot z)) \in \sim_B$. By transitive, $(s_1, x \cdot (s_2 \cdot z)) \in \sim_B$. Thus $(s_1)_{\sim_B} = (x \cdot (s_2 \cdot z))_{\sim_B}$. Since S is a UP-ideal of X, it follows from Theorem 2.7 (2) that S is a UP-filter of X. By Lemma 4.11, we have $(s_1)_{\sim_B} \subseteq S$. Thus $(x \cdot (s_2 \cdot z))_{\sim_B} \subseteq S$. Since $x \cdot (s_2 \cdot z) \in (x \cdot (s_2 \cdot z))_{\sim_B} \subseteq S$ and S is a UP-ideal of X, we have $x \cdot z \in S$. Thus $x \cdot z \in (x \cdot z)_{\sim_B} \cap S \neq \emptyset$, that is, $x \cdot z \in \sim_{B+}(S)$. Hence, $\sim_{B+}(S)$ is a UP-ideal of X. □

Example 4.24. From Example 4.18, let $S = \{0, 2, 6\}$. Then S is not a UP-ideal of X, But $\sim_{B-}(S) = \{0, 2\}$ and $\sim_{B+}(S) = \{0, 2, 3, 6\}$ are UP-ideal of X, that is, S is both a lower and an upper rough UP-ideal of X with respect to B. Hence, S is a rough UP-ideal of X with respect to B.

Theorem 4.25. *Let S be a subset of X containing a UP-ideal B of X. Then S is a strongly UP-ideal of X with respect to B if and only if S is a lower rough strongly UP-ideal of X.*

Proof. Let $x \in (0)_{\sim_B}$. Then $(x, 0) \in \sim_B$, that is, $x = 0 \cdot x \in B \subseteq S$. Thus $(0)_{\sim_B} \subseteq S$. Hence, $0 \in \sim_{B-}(S)$. Next, let $(z \cdot y) \cdot (z \cdot x) \in \sim_{B-}(S)$ and $y \in \sim_{B-}(S)$. Then $((z \cdot y) \cdot (z \cdot x))_{\sim_B} \subseteq S$ and $(y)_{\sim_B} \subseteq S$. Thus $y \in S$. We shall show that $x \in \sim_{B-}(S)$, that is, $(x)_{\sim_B} \subseteq S$. Let $a \in (x)_{\sim_B}$. Since $y \in (y)_{\sim_B}$ and $z \in (z)_{\sim_B}$, we have

$$\begin{aligned}
(z \cdot y) \cdot (z \cdot a) &\in [(z)_{\sim_B} \cdot (y)_{\sim_B}] \cdot [(z)_{\sim_B} \cdot (x)_{\sim_B}] \\
&\subseteq (z \cdot y)_{\sim_B} \cdot (z \cdot x)_{\sim_B} && \text{(By Lemma 3.6)} \\
&\subseteq ((z \cdot y) \cdot (z \cdot x))_{\sim_B} && \text{(By Lemma 3.6)} \\
&\subseteq S.
\end{aligned}$$

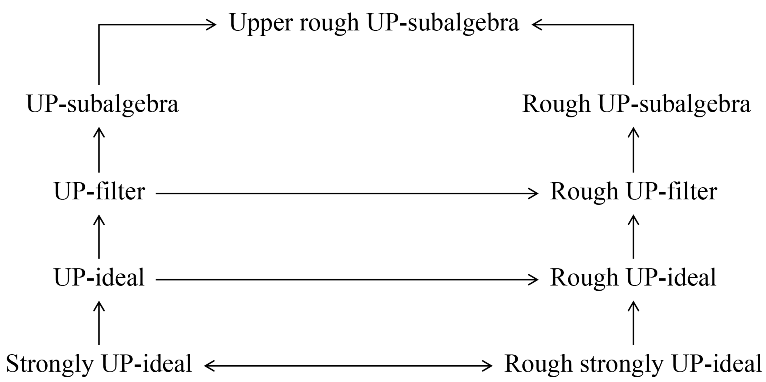
Thus $(z \cdot y) \cdot (z \cdot a) \in S$. Since S is a strongly UP-ideal of X, we have $a \in S$. Thus $(x)_{\sim_B} \subseteq S$. that is, $x \in \sim_{B-}(S)$. Hence, $\sim_{B-}(S)$ is a strongly UP-ideal of X. On the other hand, let S be a lower rough strongly UP-ideal of X. Then $\sim_{B-}(S)$ is a strongly UP-ideal of X. Thus $X = \sim_{B-}(S) \subseteq S \subseteq X$. Hence, $S = X$, it follows from Theorem 2.7 (3) that S is a strongly UP-ideal of X. □

Theorem 4.26. *Let S be a strongly UP-ideal of X containing a UP-ideal B of X. Then $\sim_{B^+}(S)$ is a strongly UP-ideal of X, that is, S is an upper rough strongly UP-ideal of X with respect to B.*

Proof. Since $0 \in (0)_{\sim_B}$ and $0 \in S$, we have $0 \in (0)_{\sim_B} \cap S \neq \emptyset$. Hence, $0 \in \sim_{B^+}(S)$. Next, let $(z \cdot y) \cdot (z \cdot x) \in \sim_{B^+}(S)$ and $y \in \sim_{B^+}(S)$. Then $((z \cdot y) \cdot (z \cdot x))_{\sim_B} \cap S \neq \emptyset$ and $(y)_{\sim_B} \cap S \neq \emptyset$. We shall show that $x \in \sim_{B^+}(S)$, that is, $(x)_{\sim_B} \cap S \neq \emptyset$. Since $((z \cdot y) \cdot (z \cdot x))_{\sim_B} \cap S \neq \emptyset$ and $(y)_{\sim_B} \cap S \neq \emptyset$ so we have $s_1, s_2 \in S$ such that $s_1 \in ((z \cdot y) \cdot (z \cdot x))_{\sim_B}$ and $s_2 \in (y)_{\sim_B}$. Thus $(s_1, (z \cdot y) \cdot (z \cdot x)) \in \sim_B$ and $(s_2, y) \in \sim_B$. Then $((z \cdot s_2) \cdot (z \cdot x), (z \cdot y) \cdot (z \cdot x)) \in \sim_B$. By transitive, we have $(s_1, (z \cdot s_2) \cdot (z \cdot x)) \in \sim_B$. Thus $(s_1)_{\sim_B} = ((z \cdot s_2) \cdot (z \cdot x))_{\sim_B}$. Since S is a strongly UP-ideal of X, we have S is a UP-filter of X. By Lemma 4.11, we have $(s_1)_{\sim_B} \subseteq S$. Thus $((z \cdot s_2) \cdot (z \cdot x))_{\sim_B} \subseteq S$. Since $(z \cdot s_2) \cdot (z \cdot x) \in ((z \cdot s_2) \cdot (z \cdot x))_{\sim_B} \subseteq S$ and S is a strongly UP-ideal of X, we have $x \in S$. Thus $x \in (x)_{\sim_B} \cap S \neq \emptyset$. Hence, $x \in \sim_{B^+}(S)$. Therefore, $\sim_{B^+}(S)$ is a strongly UP-ideal of X. □

Example 4.27. From Example 4.18, we have $S = \{0, 1, 2, 3, 5, 7\}$ is not a strongly UP-ideal of X but $\sim_{B^+}(S) = \{0, 1, 2, 3, 4, 5, 6, 7\} = X$, it follows from Theorem 2.7 (3) that $\sim_{B^+}(S)$ is a strongly UP-ideal of X, that is, S is an upper rough strongly UP-ideal of X with respect to B.

By Theorem 4.19, 4.21, 4.23 and 4.25 and Example 4.20, 4.22, and 4.24, we have that the notion of upper rough UP-subalgebras is a generalization of UP-subalgebras and rough UP-subalgebras, rough UP-filters is a generalization of UP-filters, rough UP-ideals is a generalization of UP-ideals, and rough strongly UP-ideals and strongly UP-ideals coincide. Hence, we have the following relation:



Acknowledgment

The authors wish to express their sincere thanks to the referees for the valuable suggestions which lead to an improvement of this paper.

References

[1] R. Biswas and S. Nanda, *Rough groups and rough subgroups*, Bull. Polish Acad. Sci. Math. **42** (1994), no. 3, 251–254.

[2] W. A. Dudek, Y. B. Jun, and H. S. Kim, *Rough set theory applied to BCI-algebras*, Quasigroups Relat. Syst. **9** (2002), 45–54.

[3] T. Guntasow, S. Sajak, A. Jomkham, and A. Iampan, *Fuzzy translations of a fuzzy set in UP-algebras*, J. Indones. Math. Soc. **23** (2017), no. 2, 1–19.

[4] Q. P. Hu and X. Li, *On BCH-algebras*, Math. Semin. Notes, Kobe Univ. **11** (1983), 313–320.

[5] A. Iampan, *A new branch of the logical algebra: UP-algebras*, J. Algebra Relat. Top. **5** (2017), no. 1, 35–54.

[6] Y. Imai and K. Iséki, *On axiom system of propositional calculi, XIV*, Proc. Japan Acad. **42** (1966), no. 1, 19–22.

[7] K. Iséki, *An algebra related with a propositional calculus*, Proc. Japan Acad. **42** (1966), no. 1, 26–29.

[8] Y. B. Jun, *Roughness of ideals in BCK-algebras*, Sci. Math. Japon. **7** (2002), 115–119.

[9] S. Keawrahun and U. Leerawat, *On isomorphisms of SU-algebras*, Sci. Magna **7** (2011), no. 2, 39–44.

[10] N. Kuroki, *Rough ideals in semigroups*, Inform. Sci. **100** (1997), 139–163.

[11] N. Kuroki and J. N. Mordeson, *Structure of rough sets and rough groups*, J. Fuzzy Math. **5** (1997), 183–191.

[12] Z. Pawlak, *Rough sets*, Int. J. Inform. Comp. Sci. **11** (1982), 341–356.

[13] C. Prabpayak and U. Leerawat, *On ideals and congruences in KU-algebras*, Sci. Magna **5** (2009), no. 1, 54–57.

[14] J. Somjanta, N. Thuekaew, P. Kumpeangkeaw, and A. Iampan, *Fuzzy sets in UP-algebras*, Ann. Fuzzy Math. Inform. **12** (2016), no. 6, 739–756.

BIOGRAPHY

BIOGRAPHY

Name Surname	Theeyarat Klinseesook
Date of Birth	17 December 1995
Education Background	Junior High School from Satrichaiyaphum School in 2010, Muang, Chaiyaphum, Thailand Senior High School from Satrichaiyaphum School in 2013, Muang, Chaiyaphum, Thailand
Address	House No. 18 Village No. 11 Nongkham sub-district, Khonsawan district, Chaiyaphum province 36140
Telephone Number	0933648062
E-mail	theeyarat.kss@gmail.com

BIOGRAPHY

Name Surname	Sukhontha Bukok
Date of Birth	31 December 1995
Education Background	Junior High School from Banraiwitthaya School in 2010, Banrai, Uthaithani, Thailand Senior High School from Banraiwitthaya School in 2013, Banrai, Uthaithani, Thailand
Address	House No. 16/1 Village No. 14 Khokkhwai sub-district, Banrai district, Uthaithani province 61140
Telephone Number	0984721738
E-mail	sukhonthaswitch@gmail.com

Rough Set Theory applied to UP-Algebras 2018