Convergence of a new two-step iteration scheme for two asymptotically quasi-nonexpansive type mappings in CAT(0) spaces

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Advisor and Dean of School of Science have considered the independent study entitled "Convergence of a new two-step iteration scheme for two asymptotically quasi-nonexpansive type mappings in CAT(0) spaces" submitted in partial fulfillment of the requirements for Bachelor of Science Degree in Mathematics is hereby approved.

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ชื่อเรื่อง	การลู่เข้าของระเบียบวิธีการทำซ้ำสองขั้นตอนแบบใหม่สำหรับสองการส่งชนิดไม่ขยายคล้ายแบบเชิงเส้นกำกับในปริภูมิ CAT(0)
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ในงานวิจัยฉบับนี้ ได้แนะนำและศึกษาระเบียบวิธีการทำซ้ำสองขั้นตอนแบบใหม่ สำหรับการหาจุดตรึงร่วมของสองการส่งชนิดไม่ขยายคล้ายแบบเชิงเส้นกำกับในปริภูมิ CAT(0) ที่กำหนดขึ้นโดยให้ทฤษฎีบทการลู่เข้าแบบเข้มบางอย่าง สำหรับการส่งในปริภูมิ CAT(0) ภายใต้เงื่อนไขที่เหมาะสมซึ่งผลลัพธ์ที่ได้ในงานวิจัยนี้เป็นการขยายและปรับปรุงผลลัพธ์ที่เกี่ยวข้องก่อนหน้านี้

Title	Convergence of a new two-step iteration scheme for two asymptotically quasi-nonexpansive type mappings in CAT(0) spaces
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ABSTRACT

In this paper, we introduce and study a new two-step iteration scheme for finding common fixed points of two asymptotically quasi-nonexpansive type mappings in the setting of CAT(0) spaces. We give some strong convergence theorems for such mappings in CAT(0) spaces under suitable conditions. The results obtained in this paper extend and improve the several recent results in this area.

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CHAPTER I

Introduction and Preliminaries

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert space (see [2]), $\mathbb{R}$-trees (see [12]), Euclidean buildinges (see [3]), the complex Hilbert ball with a hyperbolic metric (see [8]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [2].

Fixed point theory in a CAT(0) space has been first studied by Kirk (see [14,15]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. It is worth mentioning that the results in a CAT(0) space can be applied to any CAT(k) space with $k \leq 0$ since any CAT(k) space is a CAT(k') space for every $k' \geq k$ (see, e.g., [2]).

The Mann iteration process is defined by the sequence $\{x_n\}$,

$$(1.1) \qquad \begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \qquad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in (0,1).

Further, the Ishikawa iteration process is defined by the sequence $\{x_n\}$,

$$(1.2) \qquad \begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \qquad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in (0,1). This iteration process reduces to the Mann iteration process when $\beta_n = 0$ for all $n \geq 1$.

In 2007, Agarwal, O'Regan and Sahu [1] introduced the S-iteration process in a Banach space,

$$(1.3) \qquad \begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n, \qquad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in (0,1). They showed that their process is independent of those of Mann and Ishikawa and converges faster than both (see [1, Proposition 3.1]).

Schu [22], in 1991, considered the modified Mann iteration process which is a generalization of the Mann iteration process,

$$(1.4) \qquad \begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \qquad n \geq 1 \end{cases}$$

where $\{\alpha_n\}$ is a sequences in (0,1).

Tan and Xu [25], in 1994, studied the modified Ishikawa iteration process which is a generalization of the Ishikawa iteration process,

$$(1.5) \qquad \begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \qquad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in (0,1). This iteration process reduces to the modified Mann iteration process when $\beta_n = 0$ for all $n \geq 1$.

In 2007, Agarwal, O'Regan and Sahu [1] introduced the modified S-iteration process in a Banach space,

$$
(1.6) \qquad \begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)T^n x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \qquad n \geq 1, \end{cases}
$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in (0,1). Note that (1.6) is independent of (1.5) (and hence of (1.4)). Also (1.6) reduces to (1.3) when $T^n = T$ for all $n \geq 1$.

In 2013, Sahin and Basarir [20] modified iteration process (1.6) in a CAT(0) space as follows :

Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $T : K \to K$ be an asymptotically quasi-nonexpensive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated iteratively by

$$
(1.7) \qquad \begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)T^n x_n \bigoplus \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n \bigoplus \beta_n T^n x_n, \qquad n \geq 1, \end{cases}
$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are the sequences such that $0 \leq \alpha_n$,
$\beta_n \leq 1$ for all $n \geq 1$. They studied modified S-iteration process for asymptotically quasi-nonexpensive mappings on the CAT(0) space and established some strong convergence results under some suitable conditions which generalize some results of Khan and Abbas [10].

Very recently, Saluja [21] studied the following iteration scheme

$$
(1.8) \qquad \begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)T^n x_n \bigoplus \alpha_n S^n y_n, \\ y_n = (1-\beta_n)x_n \bigoplus \beta_n T^n x_n, \qquad n \geq 1, \end{cases}
$$

where $\{\alpha_n\}$,$\{\beta_n\}$ are the sequences such that $0 \leq \alpha_n$, $\beta_n \leq 1$ for all $n \geq 1$ and give the sufficient condition for convergence to a common fixed point in the setting of CAT(0) space and also establish some strong convergence results under some suitable conditions.

We now further modify (1.8) for two mappings in a CAT(0) space as follows: Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $S,T : K\to K$ be two asymptotically quasi-nonexpansive mappings with $F = F(S) \cap F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated iteratively by

$$\text{(1.9)} \qquad \begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)T^n y_n \bigoplus \alpha_n S^n y_n, \\ y_n = (1-\beta_n)x_n \bigoplus \beta_n T^n x_n, \qquad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$,$\{\beta_n\}$ are the sequences such that $0\leq\alpha_n$, $\beta_n\leq 1$ for all $n\geq 1$.

The aim of this paper is to study the newly defined two-step iteration scheme (1.9) for two asymptotically quasi-nonexpansive type mappings and give the sufficient condition to converge to common fixed point in the framework of CAT(0) space and also establish some strong convergence results under some suitable conditions. Our results extend and generalize many known results from the previous work given in the existing literature.

In order to prove the main results of this paper, we need the following definitions, concepts and lemmas :

Let (X,d) be a metric space and K be its nonempty subset. Let $T : K \to K$ be a mapping. A point $x \in K$ is a called a fixed point of T if $Tx = x$. We will also denote by F, The set of common fixed point of S and T, that is, $F = \{x \in K : Sx = Tx = x\}$.

The concept of quasi-nonpansive mapping was introduced by Diaz and Matcalf [5] in 1967, the concept of asymptotically nonexpansive mapping was introduced by Goebel and Kirk [7] in 1972. The iterative approximation problems for asymptotically quasi-nonexpansive mappings was studied by many authors in a Banach space and a CAT(0) space (see, e.g. [6, 11, 16, 17, 21, 23]).

DEFINITION 1.1. Let (X,d) be a metric space and K be its nonempty subset. Than $T : K \to K$ ia said to be

(1) nonexpansive if $d(Tx,Ty) \leq d(x,y)$ for all $x,y \in K$;

(2) asymptotically nonexpansive if there exists a sequence $\{r_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} r_n = 0$ such that $d(T^n x,T^n y) \leq (1 + r_n)d(x,y)$ for all $x,y \in K$ and $n \geq 1$;

(3) quasi-nonexpansive if $d(Tx,p) \leq d(x,p)$ for all $x \in K$ and $p \in F(T)$;

(4) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{r_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} r_n = 0$ such that $d(T^n x,p) \leq (1 + r_n)d(x,p)$ for all $x \in K$, $p \in F(T)$ and $n \geq 1$;

(5) asymptotically nonexpansive type [13], if

$$\limsup_{n\to\infty} \left\{ \sup_{x,y\in K} \left(d(T^n x, T^n y) - d(x,y) \right) \right\} \leq 0 \ ;$$

(6) asymptotically quasi-nonexpansive type [19], if $F(T) \neq \emptyset$ and

$$\limsup_{n\to\infty} \left\{ \sup_{x\in K, p\in F(T)} \left(d(T^n x, p) - d(x,p) \right) \right\} \leq 0 \ ;$$

(7) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$d(T^n x,T^n y) \leq Ld(x,y) \text{ for all } x,y \in K \text{ and } n \geq 1 \ ;$$

(8) semi-compact if for a sequence $\{x_n\}$ in K with $\lim_{n\to\infty} d(x_n,Tx_n) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$.

REMARK 1.2. Form Definition 2.1, it is clear that the class of quasi-nonexpansive mappings, asymptotically nonexpansive mappings, asymptotically quasi-nonexpansive mappings and asymptotically nonexpansive type mappings include nonexpansive mappings, whereas the class of asymptotically quasi-nonexpansive mappings, asymptotically nonexpansive mappings, asymptotically quasi-nonexpansive mappings and asymptotically nonexpansive type mappings. The reverse of these implications may not be true.

Let (X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t),c(t') = |t - t'|$ for all t , $t' \in [0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. We say that X is (i) a geodesic space if any two point of X are joined by a geodesic and (ii) uniquely geodesic if there is exactly one geodesic joining x and y for each $x,y \in X$, which we will denote by $[x,y]$, called the segment joining x to y.

A geodesic triangle $\triangle(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three point in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for geodesic triangle $\triangle(x_1,x_2,x_3)$ in (X,d) is a triangle $\overline{\triangle}(x_1,x_2,x_3) := \triangle(\overline{x_1},\overline{x_2},\overline{x_3})$ in Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i},\overline{x_j}) = d(x_i,x_j)$ for $i,j \in \{1,2,3\}$. Such a triangle always exists (see [2]).

CAT(0) space

A geodesic metric space is said to be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

Let $\triangle$ be a geodesic triangles in X, and let $\overline{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x.y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

(1.10) $$d(x,y) \leq d_{\mathbb{R}^2}(\overline{x},\overline{y}).$$

Complete CAT(0) space are often called Hadamard spaces (see [9]). If x,y_1,y_2 are points of a CAT(0) space and y_0 is the midpoint of the segment $[y_1,y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the CAT(0) inequality implies

(1.11) $$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2}d^2(x,y_1) + \frac{1}{2}d^2(x,y_2) - \frac{1}{4}d^2(y_1,y_2).$$

The inequality (1.10) is the (*CN*) inequality of Bruhat and Tits [4].

Let us recall that a geodesic metric space is a CAT(0) space if and only if it satisfies the (*CN*) inequality (see [2, page 163]).

A subset K of a CAT(0) space X is convex if for any $x,y \in K$, we have $[x,y] \subset K$.

We need the following useful lemmas to prove our main results in this paper.

LEMMA 1.3. (see [18]) Let X be a CAT(0) space.

(i) For $x,y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that

(A) $$d(x,z) = t\, d(x,y) \text{ and } d(y,z) = (1-t)d(x,y).$$

We use the notation $(1 - t)x \oplus ty$ for the unique point z satisfying (A).

(ii) For $x,y \in X$ and $t \in [0,1]$, we have

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z).$$

LEMMA 1.4. (see [24]) Suppose $\{a_n\}$ and $\{b_n\}$ are two sequences of nonnegative numbers such that $a_{n+1} \leq a_n + b_n$ for all $n \geq 1$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists.

EXAMPLE 1.5. (An asymptotically quasi-nonexpansive type mapping whose fixed point set is not closed). Let $X = \mathbb{R}$, $K = [-1,1]$ and $d(x,y) = |x-y|$ be the usual metric on X.

Let $T : K \to K$ be a mapping defined by

$$T(x) = \begin{cases} x, & if \quad x \in [-1,0), \\ \frac{1}{2}, & if \quad x = 0, \\ x^2, & if \quad x \in (0,1]. \end{cases}$$

Then T is an asymptotically nonexpansive type mapping and T is discontinuous at $x = 0$ and hence T is not Lipschitzian. Also notice that $F(T) = [0,1)$ is not closed. We prove that

$$(*) \qquad \limsup_{n\to\infty} \left\{ \; |T^n x - T^n y| - |x-y| \; \right\} \leq 0,$$

for all $x,y \in [-1,1]$ and $n \geq 1$. The inequality above holds trivially if $x = y = 0$ or $x,y \in [-1,0)$. Then it suffices to consider the following cases.

Case 1 ($x,y \in (0,1]$). Then

$\limsup_{n\to\infty}\Big\{\ |T^nx-T^ny|-|x-y|\ \Big\}=\limsup_{n\to\infty}\Big\{\ |x^{2n}-y^{2n}|-|x-y|\ \Big\}\leq 0.$

Case 2 ($x \in$ [-1,0) and y = 0). Then

$\limsup_{n\to\infty}\Big\{\ |T^nx-T^ny|-|x-y|\ \Big\}=\limsup_{n\to\infty}\Big\{\ |x-\frac{1}{2^n}|-|x-y|\ \Big\}\leq 0.$

Case 3 ($x \in$ [-1,0) and $y \in$ (0,1]). Then

$\limsup_{n\to\infty}\Big\{\ |T^nx-T^ny|-|x-y|\ \Big\}=\limsup_{n\to\infty}\Big\{\ |x-y^{2n}|-|x-y|\ \Big\}\leq 0.$

Case 4 (x = 0 and $y \in$ (0,1]). Then

$\limsup_{n\to\infty}\Big\{\ |T^nx-T^ny|-|x-y|\ \Big\}=\limsup_{n\to\infty}\Big\{\ |\frac{1}{2^n}-y^{2n}|-|x-y|\ \Big\}\leq 0.$

Hence the condition (*) holds. This completes the proof. ▪

Now, we give a sufficient condition which guarantees the closedness of the fixed point set of an asymptotically quasi-nonexpansive type mapping.

PROPOSITION 1.6. Let K be a nonempty subset of a complete CAT(0) space X and $T : K \to K$ be an asymptotically quasi-nonexpansive type mapping. If $G(T)$:= $\{(x,Tx) : x \in K\}$ is closed, then $F(T)$ is closed.

Proof. Let $\{p_n\}$ be a sequence in $F(T)$ such that $p_n \to p$ as $n \to \infty$. Since T is an asymptotically quasi-nonexpansive type mapping, so we put

$$c_n = \max\Big\{\ 0, \sup_{p\in F(T), n\geq 1}\Big(\ d(T^nx,p)-d(x,p)\ \Big)\ \Big\}$$

such that $\sum_{n=1}^{\infty} c_n < \infty$. Now, we have

$$\begin{aligned} d(T^np,p) &\leq d(T^np,p_n)+d(p_n,p) \\ &\leq d(p_n,p)+c_n+d(p_n,p) \\ &= 2d(p_n,p)+c_n \to 0, as \quad n\to\infty. \end{aligned}$$

Then $T^np \to p$, and so $T(T^np)$ = $T^{n+1}p \to p$. Hence by closedness of $G(T)$, $T_p = p$, that is, $F(T)$ is closed. Thus we conclude that the fixed point set for non Lipschitzian mapping must be closed. This completes the proof. ▪

CHAPTER II

Main results

In this section, we establish some strong convergence results for a new two-step iteration scheme (1.9) to converge to a common fixed point of two asymptotically quasi-nonexpansive type mapping in the setting of CAT(0) space.

THEOREM 2.1. Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $S,T : K \to K$ be two asymptotically quasi-nonexpansive type mappings with $F = F(S) \cap F(T) \neq \emptyset$ closed. Suppose that $\{x_n\}$ is defined by the iteration process (1.9). Put

$$P_n = \max\left\{ 0, \sup_{p\in F(T),n\geq 1}\left(d(S^n x,p) - d(x,p) \right) \right\} \tag{2.1}$$

and

$$Q_n = \max\left\{ 0, \sup_{p\in F(T),n\geq 1}\left(d(T^n x,p) - d(x,p) \right) \right\} \tag{2.2}$$

such that $\sum_{n=1}^{\infty} P_n < \infty$ and $\sum_{n=1}^{\infty} Q_n < \infty$. If $\liminf_{n\to\infty} d(x_n, F) = 0$ or $\limsup_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf_{p\in F} d(x, p)$, then the sequence $\{x_n\}$ converges strongly to a point of F.

Proof. Let $p \in F$. From (1.9), (2.2) and Lemma 1.3(ii), we have

$$\begin{aligned} d(y_n,p) = &\ d((1-\beta_n)x_n \oplus \beta_n T^n x_n, p) \\ \leq &\ (1-\beta_n)d(x_n,p) + \beta_n d(T^n x_n,p) \\ \leq &\ (1-\beta_n)d(x_n,p) + \beta_n[d(x_n,p) + Q_n] \\ \leq &\ d(x_n,p) + Q_n. \end{aligned} \tag{2.3}$$

Again using (1.9), (2.1)-(2.3) and Lemma 1.3(ii), we have

$$\begin{aligned} d(x_{n+1},p) = &\ d((1-\alpha_n)T^n y_n \oplus \alpha_n S^n y_n, p) \\ \leq &\ (1-\alpha_n)d(T^n y_n,p) + \alpha_n d(S^n y_n,p) \\ \leq &\ (1-\alpha_n)[d(y_n,p) + Q_n] + \alpha_n[d(y_n,p) + P_n] \\ \leq &\ (1-\alpha_n)d(y_n,p) + \alpha_n d(y_n,p) + Q_n + P_n \\ \leq &\ (1-\alpha_n)d(y_n,p) + \alpha_n[d(y_n,p) + Q_n] + Q_n + P_n \end{aligned} \tag{2.4}$$

$$
\begin{aligned}
&= d(y_n, p) + 2Q_n + P_n \\
&\le d(x_n, p) + P_n + 2Q_n \\
&= d(x_n, p) + H_n,
\end{aligned}
$$

where $H_n = P_n + 2Q_n$. Since by hypothesis $\sum_{n=1}^{\infty} P_n < \infty$ and $\sum_{n=1}^{\infty} Q_n < \infty$, it follows that $\sum_{n=1}^{\infty} H_n < \infty$.

The inequality (2.4) gives

(2.5) $$d(x_{n+1}, F) \le d(x_n, F) + H_n.$$

Since by hypothesis $\sum_{n=1}^{\infty} H_n < \infty$, by Lemma 1.4 and $\liminf_{n\to\infty} d(x_n, F) = 0$ or $\limsup_{n\to\infty} d(x_n, F) = 0$ we obtain that

(2.6) $$\lim_{n\to\infty} d(x_n, F) = 0.$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in *K*.

From (2.4), we have

(2.7)
$$
\begin{aligned}
d(x_{n+m}, p) &\le d(x_{n+m-1}, p) + H_{n+m-1} \\
&\le \ldots \\
&\le d(x_n, p) + \sum_{k=n}^{n+m-1} H_k,
\end{aligned}
$$

for the natural numbers *m,n* and $p \in F$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, therefore for any $\varepsilon > 0$. there exists a natural number n_0 such that $d(x_n, F) < \varepsilon/4$ and $\sum_{k=n}^{n+m-1} H_k < \varepsilon/4$ for all $n \ge n_0$. So, we can find $p^* \in F$ such that $d(x_{n_0}, p^*) < \varepsilon/4$. Hence, for all $n \ge n_0$ and $m \ge 1$, we have

(2.8)
$$
\begin{aligned}
d(x_{n+m}, x_n) &\le d(x_{n+m}, p^*) + d(x_n, p^*) \\
&\le d(x_{n_0}, p^*) + \sum_{k=n_0}^{n+m-1} H_k + d(x_{n_0}, p^*) + \sum_{k=n_0}^{n+m-1} H_k \\
&= 2d(x_{n_0}, p^*) + 2\sum_{k=n_0}^{n+m-1} H_k \\
&< 2(\tfrac{\varepsilon}{4}) + 2(\tfrac{\varepsilon}{4}) = \varepsilon.
\end{aligned}
$$

This shows that $\{x_n\}$ is a Cauchy sequence in *K*. Thus, the completeness of *X* implies that $\{x_n\}$ must be convergent. Assume that $\lim_{n\to\infty} x_n = q$. Since *K* is closed, therefore $q \in K$. Next, we show that $q \in F$. Now $\lim_{n\to\infty} d(x_n, F) = 0$ gives that $d(q, F) = 0$. Since *F* is closed, $q \in F$. This completes the proof. ∎

THEOREM 2.2. Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $S,T : K \to K$ be two asymptotically quasi-nonexpansive type mappings such that $F = F(S) \cap F(T) \neq \emptyset$ closed. Suppose that $\{x_n\}$ is defined by the iteration process (1.9) and P_n and Q_n be taken as in Theorem 2.1. If S and T satisfy the following conditions :

(i) $\lim_{n\to\infty} d(x_n, Sx_n) = 0$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

(ii) If the sequence $\{z_n\}$ in K satisfies $\lim_{n\to\infty} d(z_n, Sz_n) = 0$ and $\lim_{n\to\infty} d(z_n, Tz_n) = 0$, then

$$\liminf_{n\to\infty} d(z_n, F) = 0 \text{ or } \limsup_{n\to\infty} d(z_n, F) = 0.$$

Then the sequence $\{x_n\}$ converges strongly to a point of F.

Proof. If follows from the hypothesis that $\lim_{n\to\infty} d(x_n, Sx_n) = 0$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. From (ii), $\liminf_{n\to\infty} d(x_n, F) = 0$ or $\limsup_{n\to\infty} d(x_n, F) = 0$. Therefore, the sequence $\{x_n\}$ must converge to a point of F by Theorem 2.1. This completes the proof. .

THEOREM 2.3. Let K be a nonempty closed convex subset of a complete CAT(0) space X and let$S,T : K \to K$ be two uniformly L-Lipschitzian and asymptotically quasi-nonexpansive type mappings such that $F = F(S) \cap F(T) \neq \emptyset$ closed. Suppose that $\{x_n\}$ is defined by the iteration process (1.9) and P_n and Q_n be taken as in Theorem 2.1. If either S is semi-compact and $\lim_{n\to\infty} d(x_n, Sx_n) = 0$ or T is semi-compact and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then the sequence $\{x_n\}$ converges strongly to a point of F.

Proof. Suppose that T is semi-compact and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, Then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in K$. Also, we have $\lim_{j\to\infty} d(x_{n_j}, Tx_{n_j}) = 0$. Hence, we have

$$\begin{aligned} d(p, T_p) \leq & \; d(p, x_{n_j}) + d(x_{n_j}, Tx_{n_j}) + d(Tx_{n_j}, T_p) \\ \leq & \; (1+L)d(p, x_{n_j}) + d(x_{n_j}, Tx_{n_j}) \to 0. \end{aligned}$$

Thus $p \in F$. By (2.4),

$$d(x_{n+1}, p) \;\leq\; d(x_n, p) + H_n.$$

Since by hypothesis $\sum_{n=1}^{\infty} H_n < \infty$, by Lemma 1.2 and $\lim_{n\to\infty} d(x_n, p)$ exists and $x_{n_j} \to p \in F$ gives that $x_n \to p \in F$. This shows that $\{x_n\}$ converges strongly to a point of *F*. This completes the proof. ∎

As an application of Theorem 2.1, we establish another strong convergence result as follows.

THEOREM 2.4. Let *K* be a nonempty closed convex subset of a complete CAT(0) space *X* and let *S*,*T* : *K* $\to$ *K* be two asymptotically quasi-nonexpansive type mappings such that *F* = *F*(*S*) $\cap$ *F*(*T*) $\neq \emptyset$ is closed. Suppose that $\{x_n\}$ is defined by the iteration process (1.9) and P_n and Q_n be taken as in Theorem 2.1. If *S* and *T* satisfy the following conditions :

(i) $\lim_{n\to\infty} d(x_n, Sx_n) = 0$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

(ii) There exists a constant $A > 0$ such that

$$\frac{1}{2}\left[\ d(x_n, Sx_n) + d(x_n, Tx_n)\ \right] \geq Ad(x_n, F).$$

Then the sequence $\{x_n\}$ converges strongly to a point of *F*.

Proof. From conditions (i) and (ii), we have $\lim_{n\to\infty} d(x_n, F) = 0$, it follows as in the proof of Theorem 2.1, that $\{x_n\}$ must converges strongly to a point of *F*. This completes the proof. ∎

CHAPTER III

Conclusions

1) Let K be a nonempty subset of a complete CAT(0) space X and $T : K \to K$ be an asymptotically quasi-nonexpansive type mapping. If $G(T) := \{(x,Tx) : x \in K\}$ is closed, then $F(T)$ is closed.

2) Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $S,T : K \to K$ be two asymptotically quasi-nonexpansive type mappings with $F = F(S) \cap F(T) \neq \emptyset$ closed. Suppose that $\{x_n\}$ is defined by the iteration process (1.9). Put

(I) $$P_n = \max\left\{ 0, \sup_{p\in F(T),n\geq 1}\left(d(S^n x,p) - d(x,p) \right) \right\}$$

and

(II) $$Q_n = \max\left\{ 0, \sup_{p\in F(T),n\geq 1}\left(d(T^n x,p) - d(x,p) \right) \right\}$$

such that $\sum_{n=1}^{\infty} P_n < \infty$ and $\sum_{n=1}^{\infty} Q_n < \infty$. If $\liminf_{n\to\infty} d(x_n, F) = 0$ or $\limsup_{n\to\infty} d(x_n, F) = 0$, where $d(x,F) = \inf_{p\in F} d(x,p)$, then the sequence $\{x_n\}$ converges strongly to a point of F.

3) Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $S,T : K \to K$ be two asymptotically quasi-nonexpansive type mappings such that $F = F(S) \cap F(T) \neq \emptyset$ closed. Suppose that $\{x_n\}$ is defined by the iteration process (1.9) and P_n and Q_n be taken as in Theorem 2.1. If S and T satisfy the following conditions :

(i) $\lim_{n\to\infty} d(x_n, Sx_n) = 0$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

(ii) If the sequence $\{z_n\}$ in K satisfies $\lim_{n\to\infty} d(z_n, Sz_n) = 0$ and $\lim_{n\to\infty} d(z_n, Tz_n) = 0$, then

$$\liminf_{n\to\infty} d(z_n, F) = 0 \text{ or } \limsup_{n\to\infty} d(z_n, F) = 0.$$

Then the sequence $\{x_n\}$ converges strongly to a point of F.

4) Let K be a nonempty closed convex subset of a complete CAT(0) space X and let$S,T : K \to K$ be two uniformly L-Lipschitzian and asymptotically quasi-nonexpansive type mappings such that $F = F(S) \cap F(T) \neq \emptyset$ closed. Suppose that $\{x_n\}$ is defined by the iteration process (1.9) and P_n and Q_n be taken as in Theorem 2.1. If either S is semi-compact and $\lim_{n\to\infty} d(x_n, Sx_n) = 0$ or T is semi-compact and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then the sequence $\{x_n\}$ converges strongly to a point of F.

5) Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $S,T : K \to K$ be two asymptotically quasi-nonexpansive type mappings such that $F = F(S) \cap F(T) \neq \emptyset$ is closed. Suppose that $\{x_n\}$ is defined by the iteration process (1.9) and P_n and Q_n be taken as in Theorem 2.1. If S and T satisfy the following conditions :

(i) $\lim_{n\to\infty} d(x_n, Sx_n) = 0$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

(ii) There exists a constant $A > 0$ such that

$$\frac{1}{2}\left[\ d(x_n, Sx_n) + d(x_n, Tx_n)\ \right] \geq Ad(x_n, F).$$

Then the sequence $\{x_n\}$ converges strongly to a point of F.

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APPENDIX

Convergence of a new two-step iteration scheme for two asymptotically quasi-nonexpansive type mappings in CAT(0) spaces

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Abstract

In this paper, we introduce and study a new two-step iteration scheme for finding common fixed points of two asymptotically quasi-nonexpansive type mappings in the setting of CAT(0) spaces. We give some strong convergence theorems for such mappings in CAT(0) spaces under suitable conditions. The results obtained in this paper extend and generalize the several recent results in this area.

1 Introduction

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert space (see [2]), $\mathbb{R}$-trees (see [12]), Euclidean buildinges (see [3]), the complex Hilbert ball with a hyperbolic metric (see [8]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [2].

Fixed point theory in a CAT(0) space has been first studied by Kirk (see [14,15]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. It is worth mentioning that the results in a CAT(0) space can be applied to any CAT(k) space with $k \leq 0$ since any CAT(k) space is a CAT(k') space for every $k' \geq k$ (see, e.g., [2]).

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The Mann iteration process is defined by the sequence $\{x_n\}$,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \qquad n \geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in (0,1).

Further, the Ishikawa iteration process is defined by the sequence $\{x_n\}$,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \qquad n \geq 1, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in (0,1). This iteration process reduces to the Mann iteration process when $\beta_n = 0$ for all $n \geq 1$.

In 2007, Agarwal, O'Regan and Sahu [1] introduced the S-iteration process in a Banach space,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \qquad n \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in (0,1). They showed that their process is independent of those of Mann and Ishikawa and converges faster than both (see [1, Proposition 3.1]).

Schu [22], in 1991, considered the modified Mann iteration process which is a generalization of the Mann iteration process,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \qquad n \geq 1 \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ is a sequences in (0,1).

Tan and Xu [25], in 1994, studied the modified Ishikawa iteration process which is a generalization of the Ishikawa iteration process,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \qquad n \geq 1, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in (0,1). This iteration process reduces to the modified Mann iteration process when $\beta_n = 0$ for all $n \geq 1$.

In 2007, Agarwal, O'Regan and Sahu [1] introduced the modified S-iteration process in a Banach space,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)T^n x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \qquad n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in (0,1). Note that (1.6) is independent of (1.5) (and hence of (1.4)). Also (1.6) reduces to (1.3) when $T^n = T$ for all $n \geq 1$.

In 2013, Sahin and Basarir [20] modified iteration process (1.6) in a CAT(0) space as follows :

Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $T : K \to K$ be an asymptotically quasi-nonexpensive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated iteratively by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)T^n x_n \bigoplus \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n \bigoplus \beta_n T^n x_n, \qquad n \geq 1, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are the sequences such that $0 \leq \alpha_n,\beta_n \leq 1$ for all $n \geq 1$. They studied modified S-iteration process for asymptotically quasi-nonexpensive mappings on the CAT(0) space and established some strong convergence results under some suitable conditions which generalize some results of Khan and Abbas [10].

Very recently, Saluja [21] studied the following iteration scheme

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)T^n x_n \bigoplus \alpha_n S^n y_n, \\ y_n = (1-\beta_n)x_n \bigoplus \beta_n T^n x_n, \qquad n \geq 1, \end{cases} \tag{1.8}$$

where $\{\alpha_n\}$,$\{\beta_n\}$ are the sequences such that $0\leq\alpha_n$, $\beta_n\leq 1$ for all $n\geq 1$ and give the sufficient condition for convergence to a common fixed point in the setting of CAT(0) space and also establish some strong convergence results under some suitable conditions.

We now further modify (1.8) for two mappings in a CAT(0) space as follows :

Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $S,T : K\to K$ be two asymptotically quasi-nonexpansive mappings with $F = F(S) \cap F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated iteratively by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)T^n y_n \bigoplus \alpha_n S^n y_n, \\ y_n = (1-\beta_n)x_n \bigoplus \beta_n T^n x_n, \qquad n \geq 1, \end{cases} \tag{1.9}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are the sequences such that $0\leq\alpha_n$, $\beta_n\leq 1$ for all $n\geq 1$.

The aim of this paper is to study the newly defined two-step iteration scheme (1.9) for two asymptotically quasi-nonexpansive type mappings and give the sufficient condition to converge to common fixed point in the framework of CAT(0) space and also establish some strong convergence results under some suitable conditions. Our results extend and generalize many known results from the previous work given in the existing literature.

2 Preliminaries and lemmas

In order to prove the main results of this paper, we need the following definitions, concepts and lemmas :

Let (X,d) be a metric space and K be its nonempty subset. Let $T : K \to K$ be a mapping. A point $x \in K$ is a called a fixed point of T if $Tx = x$. We will also denote by F, The set of common fixed point of S and T, that is, $F = \{x \in K : Sx = Tx = x\}$.

The concept of quasi-nonpansive mapping was introduced by Diaz and Matcalf [5] in 1967, the concept of asymptotically nonexpansive mapping was introduced by Goebel and Kirk [7] in 1972. The iterative approximation problems for asymptotically quasi-nonexpansive mappings was studied by many authors in a Banach space and a CAT(0) space (see, e.g. [6, 11, 16, 17, 21, 23]).

DEFINITION 2.1. Let (X,d) be a metric space and K be its nonempty subset.
Than $T : K \to K$ ia said to be
(1) nonexpansive if $d(Tx,Ty) \leq d(x,y)$ for all $x,y \in K$;
(2) asymptotically nonexpansive if there exists a sequence $\{r_n\} \subset [0,\infty)$ with

$\lim_{n\to\infty} r_n = 0$ such that $d(T^n x, T^n y) \leq (1 + r_n)d(x,y)$ for all $x,y \in K$ and $n\geq 1$;
(3) quasi-nonexpansive if $d(Tx,p) \leq d(x,p)$ for all $x \in K$ and $p \in F(T)$;
(4) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{r_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} r_n = 0$ such that $d(T^n x,p) \leq (1 + r_n)d(x,p)$ for all $x \in K$, $p \in F(T)$ and $n\geq 1$;
(5) asymptotically nonexpansive type [13], if

$$\limsup_{n\to\infty} \Big\{ \sup_{x,y\in K} \Big(d(T^n x, T^n y) - d(x,y) \Big) \Big\} \leq 0;$$

(6) asymptotically quasi-nonexpansive type [19], if $F(T) \neq \emptyset$ and

$$\limsup_{n\to\infty} \Big\{ \sup_{x\in K, p\in F(T)} \Big(d(T^n x, p) - d(x,p) \Big) \Big\} \leq 0;$$

(7) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$d(T^n x,T^n y) \leq Ld(x,y)$ for all $x,y \in K$ and $n \geq 1$;
(8) semi-compact if for a sequence $\{x_n\}$ in K with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$,

there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$.

REMARK 2.1. Form Definition 2.1, it is clear that the class of quasi-nonexpansive mappings, asymptotically nonexpansive mappings, asymptotically quasi-nonexpansive mappings and asymptotically nonexpansive type mappings include nonexpansive mappings, whereas the class of asymptotically quasi-nonexpansive mappings, asymptotically nonexpansive mappings, asymptotically quasi-nonexpansive mappings and asymptotically nonexpansive type mappings. The reverse of these implications may not be true.

Let (X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l)$

$= y$ and $d(c(t),c(t') = |t - t'|$ for all t , $t' \in [0, l]$. In particular, c is an isometry and $d(x,y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. We say that X is (i) a geodesic space if any two point of X are joined by a geodesic and (ii) uniquely geodesic if there is exactly one geodesic joining x and y for each $x,y \in X$, which we will denote by $[x,y]$, called the segment joining x to y.

A geodesic triangle $\triangle(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three point in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for geodesic triangle $\triangle(x_1,x_2,x_3)$ in (X,d) is a triangle $\overline{\triangle}(x_1,x_2,x_3) := \triangle(\overline{x_1},\overline{x_2},\overline{x_3})$ in Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i},\overline{x_j}) = d(x_i,x_j)$ for $i,j \in \{1,2,3\}$.
Such a triangle always exists (see [2]).

CAT(0) space

A geodesic metric space is said to be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

Let $\triangle$ be a geodesic triangles in X, and let $\overline{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x.y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y}). \tag{2.1}$$

Complete CAT(0) space are often called Hadamard spaces (see [9]). If x,y_1,y_2 are points of a CAT(0) space and y_0 is the midpoint of the segment $[y_1, y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the CAT(0) inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \tag{2.2}$$

The inequality (2.1) is the (CN) inequality of Bruhat and Tits [4].

Let us recall that a geodesic metric space is a CAT(0) space if and only if it satisfies the (CN) inequality (see [2, page 163]).

A subset K of a CAT(0) space X is convex if for any $x,y \in K$, we have $[x,y] \subset K$.

We need the following useful lemmas to prove our main results in this paper.

LEMMA 2.1. (see [18]) Let X be a CAT(0) space.
(i) For $x,y \in X$ and $t \in$ [0,1], there exists a unique point $z \in [x,y]$ such that
(A) $\quad d(x,z) = t\, d(x,y)$ and $d(y,z) = (1 - t)\ d(x,y)$.

We use the notation (1 - t)$x \oplus ty$ for the unique point z satisfying (A).
(ii) For $x,y \in X$ and $t \in$ [0,1], we have

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z).$$

LEMMA 2.2. (see [24]) Suppose $\{a_n\}$ and $\{b_n\}$ are two sequences of nonnegative numbers such that $a_{n+1} \leq a_n + b_n$ for all $n \geq 1$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists.

EXAMPLE 2.1. (An asymptotically quasi-nonexpansive type mapping whose fixed point set is not closed). Let $X = \mathbb{R}$, K = [-1,1] and $d(x,y) = |x - y|$ be the usual metric on X.
Let $T : K \to K$ be a mapping defined by

$$T(x) = \begin{cases} x, & if \quad x \in [-1, 0), \\ \frac{1}{2}, & if \quad x = 0, \\ x^2, & if \quad x \in (0, 1]. \end{cases}$$

Then T is an asymptotically nonexpansive type mapping and T is discontinuous at $x = 0$ and hence T is not Lipschitzian. Also notice that $F(T)$ = [0,1) is not closed. We prove that

$$(*) \qquad \limsup_{n\to\infty} \left\{ \ |T^n x - T^n y| - |x - y| \ \right\} \leq 0,$$

for all $x,y \in$ [-1,1] and $n \geq 1$. The inequality above holds trivially if $x = y = 0$ or $x,y \in$ [-1,0). Then it suffices to consider the following cases.

Case 1 ($x,y \in$ (0,1]). Then

$$\limsup_{n\to\infty} \left\{ \ |T^n x - T^n y| - |x - y| \ \right\} = \limsup_{n\to\infty} \left\{ \ |x^{2n} - y^{2n}| - |x - y| \ \right\} \leq 0.$$

Case 2 ($x \in$ [-1,0) and $y = 0$). Then

$$\limsup_{n\to\infty} \left\{ \ |T^n x - T^n y| - |x - y| \ \right\} = \limsup_{n\to\infty} \left\{ \ |x - \tfrac{1}{2^n}| - |x - y| \ \right\} \leq 0.$$

Case 3 ($x \in$ [-1,0) and $y \in$ (0,1]). Then

$$\limsup_{n\to\infty} \left\{ \ |T^n x - T^n y| - |x - y| \ \right\} = \limsup_{n\to\infty} \left\{ \ |x - y^{2n}| - |x - y| \ \right\} \leq 0.$$

Case 4 ($x = 0$ and $y \in$ (0,1]). Then

$$\limsup_{n\to\infty} \left\{ \ |T^n x - T^n y| - |x - y| \ \right\} = \limsup_{n\to\infty} \left\{ \ |\tfrac{1}{2^n} - y^{2n}| - |x - y| \ \right\} \leq 0.$$

Hence the condition (*) holds. This completes the proof. .

Now, we give a sufficient condition which guarantees the closedness of the fixed point set of an asymptotically quasi-nonexpansive type mapping.

PROPOSITION 2.1. Let K be a nonempty subset of a complete CAT(0) space X and $T : K \to K$ be an asymptotically quasi-nonexpansive type mapping. If $G(T) := \{(x,Tx) : x \in K\}$ is closed, then $F(T)$ is closed.

Proof. Let $\{p_n\}$ be a sequence in $F(T)$ such that $p_n \to p$ as $n \to \infty$. Since T is an asymptotically quasi-nonexpansive type mapping, so we put

$$c_n = \max\left\{ \ 0, \sup\nolimits_{p\in F(T), n\geq 1} \left(\ d(T^n x, p) - d(x, p) \ \right) \ \right\}$$

such that $\sum_{n=1}^{\infty} c_n < \infty$. Now, we have

$$d(T^n p, p) \le d(T^n p, p_n) + d(p_n, p)$$

$$\le d(p_n, p) + c_n + d(p_n, p)$$

$$= 2d(p_n, p) + c_n \to 0, as \quad n \to \infty.$$

Then $T^n p \to p$, and so $T(T^n p) = T^{n+1} p \to p$. Hence by closedness of $G(T)$, $T_p = p$, that is, $F(T)$ is closed. Thus we conclude that the fixed point set for non Lipschitzian mapping must be closed. This completes the proof. ■

3 Main results

In this section, we establish some strong convergence results for a new two-step iteration scheme (1.9) to converge to a common fixed point of two asymptotically quasi-nonexpansive type mapping in the setting of CAT(0) space.

THEOREM 3.1. Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $S, T : K \to K$ be two asymptotically quasi-nonexpansive type mappings with $F = F(S) \cap F(T) \neq \emptyset$ closed. Suppose that $\{x_n\}$ is defined by the iteration process (1.9). Put

$$P_n = \max\left\{ 0, \sup_{p \in F(T), n \ge 1} \left(d(S^n x, p) - d(x, p) \right) \right\} \tag{3.1}$$

and

$$Q_n = \max\left\{ 0, \sup_{p \in F(T), n \ge 1} \left(d(T^n x, p) - d(x, p) \right) \right\} \tag{3.2}$$

such that $\sum_{n=1}^{\infty} P_n < \infty$ and $\sum_{n=1}^{\infty} Q_n < \infty$. If $\liminf_{n\to\infty} d(x_n, F) = 0$ or $\limsup_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf_{p \in F} d(x, p)$, then the sequence $\{x_n\}$ converges strongly to a point of F.

Proof. Let $p \in F$. From (1.9), (3.2) and Lemma 2.1(ii), we have

$$d(y_n, p) = d((1 - \beta_n)x_n \oplus \beta_n T^n x_n, p) \tag{3.3}$$

$$\le (1 - \beta_n) d(x_n, p) + \beta_n d(T^n x_n, p)$$

$$\le (1 - \beta_n) d(x_n, p) + \beta_n [d(x_n, p) + Q_n]$$

$$\le d(x_n, p) + Q_n.$$

Again using (1.9), (3.1)-(3.3) and Lemma 2.1(ii), we have

$$d(x_{n+1}, p) = d((1 - \alpha_n) T^n y_n \oplus \alpha_n S^n y_n, p) \tag{3.4}$$

$$\begin{aligned}
&\leq (1-\alpha_n)d(T^n y_n, p) + \alpha_n d(S^n y_n, p)\\
&\leq (1-\alpha_n)[d(y_n, p) + Q_n] + \alpha_n[d(y_n, p) + P_n]\\
&\leq (1-\alpha_n)d(y_n, p) + \alpha_n d(y_n, p) + Q_n + P_n\\
&\leq (1-\alpha_n)d(y_n, p) + \alpha_n[d(y_n, p) + Q_n] + Q_n + P_n\\
&= d(y_n, p) + 2Q_n + P_n\\
&\leq d(x_n, p) + P_n + 2Q_n\\
&= d(x_n, p) + H_n,
\end{aligned}$$

where $H_n = P_n + 2Q_n$. Since by hypothesis $\sum_{n=1}^{\infty} P_n < \infty$ and $\sum_{n=1}^{\infty} Q_n < \infty$, it follows that $\sum_{n=1}^{\infty} H_n < \infty$.

The inequality (3.4) gives

$$d(x_{n+1}, F) \leq d(x_n, F) + H_n. \tag{3.5}$$

Since by hypothesis $\sum_{n=1}^{\infty} H_n < \infty$, by Lemma 2.2 and $\liminf_{n\to\infty} d(x_n, F) = 0$ or $\limsup_{n\to\infty} d(x_n, F) = 0$ we obtain that

$$\lim_{n\to\infty} d(x_n, F) = 0. \tag{3.6}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in K.

From (3.4), we have

$$\begin{aligned}
d(x_{n+m}, p) &\leq d(x_{n+m-1}, p) + H_{n+m-1} \qquad (3.7)\\
&\leq d(x_{n+m-2}, p) + H_{n+m-2} + H_{n+m-1}\\
&\leq ...\\
&\leq d(x_n, p) + \sum_{k=n}^{n+m-1} H_k,
\end{aligned}$$

for the natural numbers m,n and $p \in F$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, therefore for any $\varepsilon > 0$. there exists a natural number n_0 such that $d(x_n, F) < \varepsilon/4$ and $\sum_{k=n}^{n+m-1} H_k < \varepsilon/4$ for all $n \geq n_0$. So, we can find $p^* \in F$ such that $d(x_{n_0}, p^*) < \varepsilon/4$. Hence, for all $n \geq n_0$ and $m \geq 1$, we have

$$\begin{aligned}
d(x_{n+m}, x_n) &\leq d(x_{n+m}, p^*) + d(x_n, p^*) \qquad (3.8)\\
&\leq d(x_{n_0}, p^*) + \sum_{k=n_0}^{n+m-1} H_k + d(x_{n_0}, p^*) + \sum_{k=n_0}^{n+m-1} H_k\\
&= 2d(x_{n_0}, p^*) + 2\sum_{k=n_0}^{n+m-1} H_k\\
&< 2(\frac{\varepsilon}{4}) + 2(\frac{\varepsilon}{4}) = \varepsilon.
\end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence in K. Thus, the completeness of X implies that $\{x_n\}$ must be convergent. Assume that $\lim_{n\to\infty} x_n = q$. Since K is closed, therefore $q \in K$. Next, we show that $q \in F$. Now $\lim_{n\to\infty} d(x_n, F) = 0$ gives that $d(q, F) = 0$. Since F is closed, $q \in F$. This completes the proof. ▪

THEOREM 3.2. Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $S, T : K \to K$ be two asymptotically quasi-nonexpansive type mappings such that $F = F(S) \cap F(T) \neq \emptyset$ closed. Suppose that $\{x_n\}$ is defined by the iteration process (1.9) and P_n and Q_n be taken as in Theorem 3.1. If S and T satisfy the following conditions :

(i) $\lim_{n\to\infty} d(x_n, Sx_n) = 0$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

(ii) If the sequence $\{z_n\}$ in K satisfies $\lim_{n\to\infty} d(z_n, Sz_n) = 0$ and $\lim_{n\to\infty} d(z_n, Tz_n) = 0$, then $\liminf_{n\to\infty} d(z_n, F) = 0$ or $\limsup_{n\to\infty} d(z_n, F) = 0$.

Then the sequence $\{x_n\}$ converges strongly to a point of F.

Proof. If follows from the hypothesis that $\lim_{n\to\infty} d(x_n, Sx_n) = 0$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. From (ii), $\liminf_{n\to\infty} d(x_n, F) = 0$ or $\limsup_{n\to\infty} d(x_n, F) = 0$. Therefore, the sequence $\{x_n\}$ must converge to a point of F by Theorem 3.1. This completes the proof. ▪

THEOREM 3.3. Let K be a nonempty closed convex subset of a complete CAT(0) space X and let$S, T : K \to K$ be two uniformly L-Lipschitzian and asymptotically quasi-nonexpansive type mappings such that $F = F(S) \cap F(T) \neq \emptyset$ closed. Suppose that $\{x_n\}$ is defined by the iteration process (1.9) and P_n and Q_n be taken as in Theorem 3.1. If either S is semi-compact and $\lim_{n\to\infty} d(x_n, Sx_n) = 0$ or T is semi-compact and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then the sequence $\{x_n\}$ converges strongly to a point of F.

Proof. Suppose that T is semi-compact and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, Then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in K$. Also, we have $\lim_{j\to\infty} d(x_{n_j}, Tx_{n_j}) = 0$. Hence, we have

$$\begin{aligned} d(p, Tp) &\leq d(p, x_{n_j}) + d(x_{n_j}, Tx_{n_j}) + d(Tx_{n_j}, Tp) \\ &\leq (1 + L)d(p, x_{n_j}) + d(x_{n_j}, Tx_{n_j}) \to 0. \end{aligned}$$

Thus $p \in F$. By (3.4),

$$d(x_{n+1}, p) \leq d(x_n, p) + H_n.$$

Since by hypothesis $\sum_{n=1}^{\infty} H_n < \infty$, by Lemma 2.2 and $\lim_{n\to\infty} d(x_n, p)$ exists and $x_{n_j} \to p \in F$ gives that $x_n \to p \in F$. This shows that $\{x_n\}$ converges strongly to a point of F. This completes the proof. ▪

As an application of Theorem 3.1, we establish another strong convergence result as follows.

THEOREM 3.4. Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $S, T : K \to K$ be two asymptotically quasi-nonexpansive type mappings such that $F = F(S) \cap F(T) \neq \emptyset$ is closed. Suppose that $\{x_n\}$ is defined by the iteration process (1.9) and P_n and Q_n be taken as in Theorem 3.1. If S and T satisfy the following conditions :

(i) $\lim_{n\to\infty} d(x_n, Sx_n) = 0$ and$\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

(ii) There exists a constant $A > 0$ such that

$$\frac{1}{2}\Big[\ d(x_n, Sx_n) + d(x_n, Tx_n)\ \Big] \geq Ad(x_n, F).$$

Then the sequence $\{x_n\}$ converges strongly to a point of F.

Proof. From conditions (i) and (ii), we have $\lim_{n\to\infty} d(x_n, F) = 0$, it follows as in the proof of Theorem 3.1, that $\{x_n\}$ must converges strongly to a point of F. This completes the proof. ▪

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