GENERALIZED FUZZY SETS IN UP-ALGEBRAS

NOPPHARAT DOKKHAMDANG
AKEKARIN KESORN

An Independent Study Submitted in Partial Fulfillment
of the Requirements for the degree of Bachelor
of Science Program in Mathematics
April 2018
University of Phayao

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Advisor and Dean of School of Science have considered the independent study entitled "Generalized Fuzzy Sets in UP-Algebras" submitted in partial fulfillment of the requirements for the degree of Bachelor of Science Program in Mathematics is hereby approved

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ชื่อเรื่อง	เซตวิภัชนัยวางนัยทั่วไปในพีชคณิตยูพี
ผู้ศึกษาค้นคว้า	นายนพรัตน์ ดอกคำแดง นายเอกรินทร์ เกษร
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บทคัดย่อ

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ABSTRACT

Based on the theory of fuzzy sets, the notion of fuzzy UP-subalgebras (resp., fuzzy UP-filters, fuzzy UP-ideals, fuzzy strongly UP-ideals) with thresholds of UP-algebras is introduced, some properties of them are discussed, and its generalizations are proved. Further, we discuss the relations between fuzzy UP-subalgebras (resp., fuzzy UP-filters, fuzzy UP-ideals, fuzzy strongly UP-ideals) with thresholds and its level subsets.

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CHAPTER 1
Introduction

A fuzzy set in a nonempty set X is an arbitrary function from the set X into $[0, 1]$ where $[0, 1]$ is the unit segment of the real line. The concept of a fuzzy set in a nonempty set was first considered by Zadeh [14] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere. There are several kinds of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, bipolar-valued fuzzy sets etc.

After the introduction of the notion of fuzzy sets by Zadeh [14], several researches were conducted on the generalizations of the notion of fuzzy sets and application to many logical algebras such as: In 2005, Jun [4] introduced the notion of (α, β)-fuzzy subalgebras of BCK/BCI-algebras. In 2007, Jun [5] introduced the notion of fuzzy subalgebras with thresholds of BCK/BCI-algebras. In 2009, Saeid [11] introduced the notion of new fuzzy subalgebras with thresholds of BCK/BCI-algebras. Zhan et al. [15] introduced the notions of $(\in, \in \vee q)$-fuzzy p-ideals, $(\in, \in \vee q)$-fuzzy q-ideals and $(\in, \in \vee q)$-fuzzy a-ideals in BCI-algebras. In 2013, Li and Sun [7] introduced the notion of intuitionistic fuzzy implicative ideals with thresholds (λ, μ) of BCI-algebras. Zulfiqar [16] introduced the notion of fuzzy fantastic ideals in BCH-algebras. In 2014, Sun and Li [13] introduced the notion of intuitionistic fuzzy subalgebras with thresholds (λ, μ) and intuitionistic fuzzy ideals with thresholds (λ, μ) of BCI-algebras. Zulfiqar [17] introduced the notion of $(\overline{\alpha}, \overline{\beta})$-fuzzy fantastic ideals in BCH-algebras.

In this paper, we introduce the notion of fuzzy UP-subalgebras (resp., fuzzy UP-filters, fuzzy UP-ideals, fuzzy strongly UP-ideals) with thresholds of UP-algebras, investigate their properties, and generalize the notions. Also, the characterizations of fuzzy UP-subalgebras (resp., fuzzy UP-filters, fuzzy UP-ideals, fuzzy strongly UP-ideals) with thresholds are obtained. Further, we discuss the relations between fuzzy UP-subalgebras (resp., fuzzy UP-filters, fuzzy UP-ideals, fuzzy strongly UP-ideals) with thresholds and its level subsets.

CHAPTER 2
Basic Results on UP-Algebras

2.1 Basic Results on UP-Algebras

Before we begin our study, we will introduce the definition of a UP-algebra.

An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* [2] where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms: for any $x, y, z \in A$,

(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,

(UP-2) $0 \cdot x = x$,

(UP-3) $x \cdot 0 = 0$, and

(UP-4) $x \cdot y = 0$ and $y \cdot x = 0$ imply $x = y$.

From [2], we know that the notion of UP-algebras is a generalization of KU-algebras.

Example 2.1.1. [2] Let X be a universal set. Define two binary operations $\cdot$ and $*$ on the power set of X by putting $A \cdot B = B \cap A'$ and $A * B = B \cup A'$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X), \cdot, \emptyset)$ and $(\mathcal{P}(X), *, X)$ are UP-algebras and we shall call it the *power UP-algebra of type 1* and the *power UP-algebra of type 2*, respectively.

In what follows, let A be a UP-algebra unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 2.1.2. *[2] In a UP-algebra A, the following properties hold: for any $x, y, z \in A$,*

(1) $x \cdot x = 0$,

(2) $x \cdot y = 0$ *and* $y \cdot z = 0$ *imply* $x \cdot z = 0$,

(3) $x \cdot y = 0$ *implies* $(z \cdot x) \cdot (z \cdot y) = 0$*,*

(4) $x \cdot y = 0$ *implies* $(y \cdot z) \cdot (x \cdot z) = 0$*,*

(5) $x \cdot (y \cdot x) = 0$*,*

(6) $(y \cdot x) \cdot x = 0$ *if and only if* $x = y \cdot x$*, and*

(7) $x \cdot (y \cdot y) = 0$.

Definition 2.1.3. [2] A subset S of A is called a *UP-subalgebra* of A if the constant 0 of A is in S, and $(S, \cdot, 0)$ itself forms a UP-algebra.

Iampan [2] proved the useful criteria that a nonempty subset S of a UP-algebra $A = (A, \cdot, 0)$ is a UP-subalgebra of A if and only if S is closed under the $\cdot$ multiplication on A.

Definition 2.1.4. [2, 12] A subset S of A is called a

(1) *UP-ideal* of A if it satisfies the following properties:

  (1) the constant 0 of A is in S, and

  (2) for any $x, y, z \in A, x \cdot (y \cdot z) \in S$ and $y \in S$ imply $x \cdot z \in S$.

(2) *UP-filter* of A if it satisfies the following properties:

  (1) the constant 0 of A is in S, and

  (2) for any $x, y \in A, x \cdot y \in S$ and $x \in S$ imply $y \in S$.

(3) *strongly UP-ideal* of A if it satisfies the following properties:

  (1) the constant 0 of A is in S, and

  (2) for any $x, y, z \in A, (z \cdot y) \cdot (z \cdot x) \in S$ and $y \in S$ imply $x \in S$.

Guntasow et al. [1] proved the generalization that the notion of UP-subalgebras is a generalization of UP-filters, the notion of UP-filters is a generalization of UP-ideals, and the notion of UP-ideals is a generalization of strongly UP-ideals. Moreover, they also proved that a UP-algebra A is the only one strongly UP-ideal of itself.

Definition 2.1.5. [14] A *fuzzy set* in a nonempty set X (or a fuzzy subset of X) is an arbitrary function from the set X into $[0, 1]$ where [0,1] is the unit segment of the real line.

Somjanta et al. [12] introduced the notion of fuzzy UP-subalgebras (resp., fuzzy UP-filters, fuzzy UP-ideals, fuzzy strongly UP-ideals) of UP-algebras as follows:

Definition 2.1.6. A fuzzy set f in A is called a

(1) *fuzzy UP-subalgebra* of A if for any $x, y \in A, f(x \cdot y) \geq \min\{f(x), f(y)\}$.

(2) *fuzzy UP-filter* of A if for any $x, y \in A$,

  (1) $f(0) \geq f(x)$, and

  (2) $f(y) \geq \min\{f(x \cdot y), f(x)\}$.

(3) *fuzzy UP-ideal* of A if for any $x, y, z \in A$,

  (1) $f(0) \geq f(x)$, and

  (2) $f(x \cdot z) \geq \min\{f(x \cdot (y \cdot z)), f(y)\}$.

(4) *fuzzy strongly UP-ideal* of A if for any $x, y, z \in A$,

  (1) $f(0) \geq f(x)$, and

  (2) $f(x) \geq \min\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$.

They also proved that the notion of fuzzy UP-subalgebras is a generalization of fuzzy UP-filters, the notion of fuzzy UP-filters is a generalization of fuzzy UP-ideals, and the notion of fuzzy UP-ideals is a generalization of fuzzy strongly UP-ideals.

Definition 2.1.7. [8] Let X and Y be any two nonempty sets and let $f : X \to Y$ be any function. A fuzzy set μ in X is called *f-invariant* if $f(x) = f(y)$ implies $\mu(x) = \mu(y)$ for all $x, y \in X$.

Definition 2.1.8. [2] Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras. A mapping f from A to B is called a *UP-homomorphism* if

$$f(x \cdot y) = f(x) * f(y) \text{ for all } x, y \in A.$$

A UP-homomorphism $f \colon A \to B$ is called a

(1) *UP-endomorphism* of A if $B = A$,

(2) *UP-epimorphism* if it is surjective,

(3) *UP-monomorphism* if it is injective, and

(4) *UP-isomorphism* if it is bijective.

Iampan [2] proved that $f(0_A) = 0_B$.

CHAPTER 3
Main Results

3.1 Fuzzy UP-Subalgebras (resp., Fuzzy UP-Filters, Fuzzy UP-Ideals, Fuzzy Strongly UP-Ideals) with Thresholds

In this section, we introduce the notion of fuzzy UP-subalgebras (resp., fuzzy UP-filters, fuzzy UP-ideals, fuzzy strongly UP-ideals) with thresholds of UP-algebras, investigate their properties, and generalize the notions.

Definition 3.1.1. A fuzzy set f in A is called a *fuzzy UP-subalgebra with thresholds* ε *and* δ of A where $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$ if for any $x, y \in A$,

$$\max\{f(x \cdot y), \varepsilon\} \geq \min\{f(x), f(y), \delta\}.$$

Example 3.1.2. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	3	0
2	0	1	0	0
3	0	1	3	0

Then $(A, \cdot, 0)$ is a UP-algebra. We defined a fuzzy set $f\colon A \to [0, 1]$ as follows:

$$f(0) = 0.6, f(1) = 0.7, f(2) = 0.3, \text{ and } f(3) = 0.2.$$

Then f is a fuzzy UP-subalgebra with thresholds $\varepsilon = 0.8$ and $\delta = 0.9$ of A.

Lemma 3.1.3. *If f is a fuzzy UP-subalgebra with thresholds ε and δ of A, then* $\max\{f(0), \varepsilon\} \geq \min\{f(x), \delta\}$ *for all $x \in A$. In particular, if there exists $y \in A$ such that $f(y) > \delta$, then $f(0) > \varepsilon$.*

Proof. For all $x \in A$,

$$\begin{aligned} \max\{f(0), \varepsilon\} &= \max\{f(x \cdot x), \varepsilon\} && \text{(Proposition 2.1.2 (1))} \\ &\geq \min\{f(x), f(x), \delta\} \\ &= \min\{f(x), \delta\}. \end{aligned}$$

If there exists $y \in A$ such that $f(y) > \delta$, then

$$\max\{f(0), \varepsilon\} \geq \min\{f(y), \delta\} = \delta > \varepsilon.$$

Thus $f(0) = \max\{f(0), \varepsilon\} > \varepsilon$. □

Definition 3.1.4. A fuzzy set f in A is called a *fuzzy UP-filter with thresholds ε and δ of A* where $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$ if it satisfies the following properties: for any $x, y \in A$,

(1) $\max\{f(0), \varepsilon\} \geq \min\{f(x), \delta\}$, and

(2) $\max\{f(y), \varepsilon\} \geq \min\{f(x \cdot y), f(x), \delta\}$.

Example 3.1.5. Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	0	0
2	0	1	0	0	0
3	0	1	2	0	4
4	0	1	2	3	0

Then $(A, \cdot, 0)$ is a UP-algebra. We defined a fuzzy set $f : A \to [0, 1]$ as follows:

$$f(0) = 0.9, f(1) = 0.1, f(2) = 0.5, f(3) = 0.4, \text{ and } f(4) = 0.4.$$

Then f is a fuzzy UP-filter with thresholds $\varepsilon = 0.8$ and $\delta = 0.9$ of A.

Definition 3.1.6. A fuzzy set f in A is called a *fuzzy UP-ideal with thresholds ε and δ of A* where $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$, if it satisfies the following properties: for any $x, y, z \in A$,

(1) $\max\{f(0), \varepsilon\} \geq \min\{f(x), \delta\}$, and

(2) $\max\{f(x \cdot z), \varepsilon\} \geq \min\{f(x \cdot (y \cdot z)), f(y), \delta\}$.

Example 3.1.7. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	0	1	0	0
3	0	1	3	0

Then $(A, \cdot, 0)$ is a UP-algebra. We defined a fuzzy set $f\colon A \to [0, 1]$ as follows:

$$f(0) = 0, f(1) = 0.2, f(2) = 0.1, \text{ and } f(3) = 0.3.$$

Then f is a fuzzy UP-ideal with thresholds $\varepsilon = 0.4$ and $\delta = 0.9$ of A.

Definition 3.1.8. A fuzzy set f in A is called a *fuzzy strongly UP-ideal with thresholds ε and δ of* A where $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$ if it satisfies the following properties: for any $x, y, z \in A$,

(1) $\max\{f(0), \varepsilon\} \geq \min\{f(x), \delta\}$, and

(2) $\max\{f(x), \varepsilon\} \geq \min\{f((z \cdot y) \cdot (z \cdot x)), f(y), \delta\}$.

Example 3.1.9. Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	0	4
2	0	1	0	0	0
3	0	1	2	0	4
4	0	1	2	3	0

Then $(A, \cdot, 0)$ is a UP-algebra. We defined a fuzzy set $f\colon A \to [0, 1]$ as follow:

$$f(0) = 0.3, f(1) = 0.4, f(2) = 0, f(3) = 0.1, \text{ and } f(4) = 0.$$

Then f is a fuzzy strongly UP-ideal with thresholds $\varepsilon = 0.5$ and $\delta = 0.8$ of A.

Theorem 3.1.10. *Every fuzzy strongly UP-ideal with thresholds ε and δ of A is a fuzzy UP-ideal with thresholds ε and δ.*

Proof. Assume that f is a fuzzy strongly UP-ideal with thresholds ε and δ of A. Then $\max\{f(0), \varepsilon\} \geq \min\{f(x), \delta\}$ for all $x \in A$. Let $x, y, z \in A$.

Case 1: $f(0) \leq \varepsilon$. Then $f(x) \leq \varepsilon$ for all $x \in A$. Thus

$$\begin{aligned}\max\{f(x \cdot z), \varepsilon\} &= \varepsilon \\ &\geq f(y) \\ &\geq \min\{f(x \cdot (y \cdot z)), f(y), \delta\}.\end{aligned}$$

Case 2: $f(0) > \varepsilon$. Then

$$\begin{aligned}\max\{f(x \cdot z), \varepsilon\} &\geq \min\{f((z \cdot y) \cdot (z \cdot (x \cdot z))), f(y), \delta\} \\ &= \min\{f((z \cdot y) \cdot 0), f(y), \delta\} && \text{(Proposition 2.1.2 (5))} \\ &= \min\{f(0), f(y), \delta\}. && \text{((UP-3))}\end{aligned}$$

If $\min\{f(0), f(y), \delta\} = f(y)$ or δ, we obtain immediately that $\max\{f(x \cdot z), \varepsilon\} \geq \min\{f(x \cdot (y \cdot z)), f(y), \delta\}$. Assume that $\min\{f(0), f(y), \delta\} = f(0)$. Then

$$\begin{aligned}\min\{f(0), f(y), \delta\} &= f(0) \\ &= \max\{f(0), \varepsilon\} \\ &\geq \min\{f(y), \delta\} \\ &\geq \min\{f(x \cdot (y \cdot z)), f(y), \delta\}.\end{aligned}$$

Hence, f is a fuzzy UP-ideal with thresholds ε and δ of A. □

Example 3.1.11. Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	0	4
2	0	1	0	0	0
3	0	1	2	0	4
4	0	1	2	3	0

Then $(A, \cdot, 0)$ is a UP-algebra. We defined a fuzzy set $f\colon A \to [0,1]$ as follow:

$$f(0) = 1, f(1) = 0.2, f(2) = 0.1, f(3) = 0.2, \text{ and } f(4) = 0.9.$$

Then f is a fuzzy UP-ideal with thresholds $\varepsilon = 0.5$ and $\delta = 0.9$ of A. Since $\max\{f(2), \varepsilon\} = \max\{0.1, 0.5\} = 0.5 \ngeq 0.9 = \min\{1, 1, 0.9\} = \min\{f((2 \cdot 0) \cdot (2 \cdot 2)), f(0), \delta\}$, we have f is not a fuzzy strongly UP-ideal with thresholds $\varepsilon = 0.5$ and $\delta = 0.9$ of A.

Theorem 3.1.12. *Every fuzzy UP-ideal with thresholds ε and δ of A is a fuzzy UP-filter with thresholds ε and δ.*

Proof. Assume that f is a fuzzy UP-ideal with thresholds ε and δ of A. Then $\max\{f(0), \varepsilon\} \geq \min\{f(x), \delta\}$ for all $x \in A$. Let $x, y \in A$. Then

$$\begin{aligned} \max\{f(y), \varepsilon\} &= \max\{f(0 \cdot y), \varepsilon\} && \text{((UP-2))} \\ &\geq \min\{f(0 \cdot (x \cdot y)), f(x), \delta\} \\ &= \min\{f(x \cdot y), f(x), \delta\}. && \text{((UP-2))} \end{aligned}$$

Hence, f is a fuzzy UP-filter with thresholds ε and δ of A. □

Example 3.1.13. Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	3
3	0	1	2	0	3
4	0	1	2	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We defined a fuzzy set $f\colon A \to [0,1]$ as follow:

$$f(0) = 0.9, f(1) = 0.8, f(2) = 0.7, f(3) = 0.5, \text{ and } f(4) = 0.5.$$

Then f is a fuzzy UP-filter with thresholds $\varepsilon = 0.2$ and $\delta = 0.9$ of A. Since $\max\{f(3 \cdot 4), \varepsilon\} = \max\{0.5, 0.2\} = 0.5 \ngeq 0.7 = \min\{0.9, 0.7, 0.9\} = \min\{f(3 \cdot (2 \cdot 4)), f(2), \delta\}$, we have f is not a fuzzy UP-ideal with thresholds $\varepsilon = 0.2$ and $\delta = 0.9$ of A.

Theorem 3.1.14. *Every fuzzy UP-filter with thresholds ε and δ of A is a fuzzy UP-subalgebra with thresholds ε and δ.*

Proof. Assume that f is a fuzzy UP-filter with thresholds ε and δ of A. Then $\max\{f(0), \varepsilon\} \geq \min\{f(x), \delta\}$ for all $x \in A$. Let $x, y \in A$. Then

Case 1: $f(0) \leq \varepsilon$. Then $f(x) \leq \varepsilon$ for all $x \in A$. Thus

$$\begin{aligned}\max\{f(x \cdot y), \varepsilon\} &= \varepsilon \\ &\geq f(x) \\ &\geq \min\{f(x), f(y), \delta\}.\end{aligned}$$

Case 2: $f(0) > \varepsilon$. Then

$$\begin{aligned}\max\{f(x \cdot y), \varepsilon\} &\geq \min\{f(y \cdot (x \cdot y)), f(y), \delta\} \\ &= \min\{f(0), f(y), \delta\}. && \text{(Proposition 2.1.2 (5))}\end{aligned}$$

If $\min\{f(0), f(y), \delta\} = f(y)$ or δ, we obtain immediately that $\max\{f(x \cdot y), \varepsilon\} \geq \min\{f(x), f(y), \delta\}$. Assume that $\min\{f(0), f(y), \delta\} = f(0)$. Then

$$\begin{aligned}\min\{f(0), f(y), \delta\} &= f(0) \\ &= \max\{f(0), \varepsilon\} \\ &\geq \min\{f(y), \delta\} \\ &\geq \min\{f(x), f(y), \delta\}.\end{aligned}$$

Hence, f is a fuzzy UP-subalgebra with thresholds ε and δ of A. □

Example 3.1.15. Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	3
3	0	1	2	0	3
4	0	1	2	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We defined a fuzzy set $f\colon A \to [0, 1]$ as follow:

$$f(0) = 0.8, f(1) = 0.7, f(2) = 0.4, f(3) = 0.3, \text{ and } f(4) = 0.2.$$

Then f is a fuzzy UP-subalgebra with thresholds $\varepsilon = 0.2$ and $\delta = 0.9$ of A. Since $\max\{f(4), \varepsilon\} = \max\{0.2, 0.2\} = 0.2 \ngeq 0.3 = \min\{0.3, 0.3, 0.9\} = \min\{f(3 \cdot 4), f(3), \delta\}$, we have f is not a fuzzy UP-filter with thresholds $\varepsilon = 0.2$ and $\delta = 0.9$ of A.

By Theorem 3.1.10, 3.1.12, and 3.1.14 and Example 3.1.11, 3.1.13, and 3.1.15, we have that the notion of fuzzy UP-subalgebras with thresholds ε and δ is a generalization of fuzzy UP-filters with thresholds ε and δ, the notion of fuzzy UP-filters with thresholds ε and δ is a generalization of fuzzy UP-ideals with thresholds ε and δ, and the notion of fuzzy UP-ideals with thresholds ε and δ is a generalization of fuzzy strongly UP-ideals with thresholds ε and δ.

Theorem 3.1.16. *Let $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. If a fuzzy set f in A is constant, then it is a fuzzy strongly UP-ideal with thresholds ε and δ of A.*

Proof. Assume that f is a constant fuzzy set in A and let $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. Then $f(x) = f(0)$ for all $x \in A$. Let $x \in A$.

$$\max\{f(0), \varepsilon\} \geq f(0) = f(x) \geq \min\{f(x), \delta\}.$$

Let $x, y, z \in A$. Then

$$\begin{aligned} \max\{f(x), \varepsilon\} &\geq f(x) \\ &= \min\{f((z \cdot y) \cdot (z \cdot x)), f(y)\} \\ &\geq \min\{f((z \cdot y) \cdot (z \cdot x)), f(y), \delta\}. \end{aligned}$$

Hence, f is a fuzzy strongly UP-ideal with thresholds ε and δ of A. □

Theorem 3.1.17. *Let $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. If a fuzzy set f in A is such that $f(x) \leq \varepsilon$ for all $x \in A$, then it is a fuzzy strongly UP-ideal (resp. fuzzy UP-ideal, fuzzy UP-filter and fuzzy UP-subalgebra) with thresholds ε and δ of A.*

Proof. (1) Let $x \in A$. Then

$$\max\{f(0), \varepsilon\} = \varepsilon \geq f(x) = \min\{f(x), \delta\}.$$

Let $x, y, z \in A$. Then

$$\begin{aligned}\max\{f(x), \varepsilon\} &= \varepsilon \\ &\geq \min\{f((z \cdot y) \cdot (z \cdot x)), f(y)\} \\ &= \min\{f((z \cdot y) \cdot (z \cdot x)), f(y), \delta\}.\end{aligned}$$

Hence, f is a fuzzy strongly UP-ideal with thresholds ε and δ of A.

(2) Let $x, y, z \in A$. Then

$$\begin{aligned}\max\{f(x \cdot z), \varepsilon\} &= \varepsilon \\ &\geq \min\{f(x \cdot (y \cdot z)), f(y)\} \\ &= \min\{f(x \cdot (y \cdot z)), f(y), \delta\}.\end{aligned}$$

Hence, f is a fuzzy UP-ideal with thresholds ε and δ of A.

(3) Let $x, y \in A$. Then

$$\begin{aligned}\max\{f(y), \varepsilon\} &= \varepsilon \\ &\geq \min\{f(x \cdot y), f(x)\} \\ &= \min\{f(x \cdot y), f(x), \delta\}.\end{aligned}$$

Hence, f is a fuzzy UP-filter with thresholds ε and δ of A.

(4) Let $x, y \in A$. Then

$$\begin{aligned}\max\{f(x \cdot y), \varepsilon\} &= \varepsilon \\ &\geq \min\{f(x), f(y)\} \\ &= \min\{f(x), f(y), \delta\}.\end{aligned}$$

Hence, f is a fuzzy UP-subalgebra with thresholds ε and δ of A. □

Theorem 3.1.18. *Let $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. If a fuzzy set f in A with $\varepsilon \leq f(x) \leq \delta$ is a fuzzy strongly UP-ideal with thresholds ε and δ of A, then it is constant.*

Proof. For all $x \in A$,

$$f(0) = \max\{f(0), \varepsilon\} \geq \min\{f(x), \delta\} = f(x)$$

and

$$\begin{aligned} f(x) &= \max\{f(x), \varepsilon\} \\ &\geq \min\{f((x \cdot 0) \cdot (x \cdot x)), f(0), \delta\} \\ &= \min\{f((x \cdot 0) \cdot 0), f(0), \delta\} && \text{(Proposition 2.1.2 (1))} \\ &= \min\{f(0), f(0), \delta\} && \text{((UP-3))} \\ &= \min\{f(0), \delta\} \\ &= f(0). \end{aligned}$$

Hence, $f(x) = f(0)$ for all $x \in A$, so f is constant. □

Theorem 3.1.19. *Let $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. If a fuzzy set f in A is such that $f(x) \geq \delta$ for all $x \in A$, then it is a fuzzy strongly UP-ideal (resp. fuzzy UP-ideal, fuzzy UP-filter and fuzzy UP-subalgebra) with thresholds ε and δ of A.*

Proof. (1) Let $x \in A$. Then

$$\max\{f(0), \varepsilon\} = f(0) \geq \delta = \min\{f(x), \delta\}.$$

Let $x, y, z \in A$. Then

$$\begin{aligned} \max\{f(x), \varepsilon\} &= f(x) \\ &\geq \delta \\ &= \min\{f((z \cdot y) \cdot (z \cdot x)), f(y), \delta\}. \end{aligned}$$

Hence, f is a fuzzy strongly UP-ideal with thresholds ε and δ of A.

(2) Let $x, y, z \in A$. Then

$$\begin{aligned} \max\{f(x \cdot z), \varepsilon\} &= f(x \cdot z) \\ &\geq \delta \\ &= \min\{f(x \cdot (y \cdot z)), f(y), \delta\}. \end{aligned}$$

Hence, f is a fuzzy UP-ideal with thresholds ε and δ of A.

(3) Let $x, y \in A$. Then

$$\begin{aligned}\max\{f(y), \varepsilon\} &= f(y) \\ &\geq \delta \\ &= \min\{f(x \cdot y), f(x), \delta\}.\end{aligned}$$

Hence, f is a fuzzy UP-filter with thresholds ε and δ of A.

(4) Let $x, y \in A$. Then

$$\begin{aligned}\max\{f(x \cdot y), \varepsilon\} &= f(x \cdot y) \\ &\geq \delta \\ &= \min\{f(x), f(y), \delta\}.\end{aligned}$$

Hence, f is a fuzzy UP-subalgebra with thresholds ε and δ of A. □

Corollary 3.1.20. *Let $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. Then a fuzzy set f in A with $\varepsilon \leq f(x) \leq \delta$ is a fuzzy strongly UP-ideal with thresholds ε and δ of A if and only if it is constant.*

Proof. It is straightforward by Theorem 3.1.16 and 3.1.18. □

3.2 Upper t-Level Subsets of a Fuzzy Set

In this section, we discuss the relations between fuzzy UP-subalgebras (resp., fuzzy UP-filters, fuzzy UP-ideals, fuzzy strongly UP-ideals) with thresholds and its level subsets.

Let f be a fuzzy set in A. For any $t \in [0, 1]$, the set

$$U(f; t) = \{x \in A \mid f(x) \geq t\}$$

is called an *upper t-level subset* [12] of f.

Theorem 3.2.1. *Let f be a fuzzy set in A and $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. Then f is a fuzzy UP-subalgebra with thresholds ε and δ of A if and only if for all $t \in (\varepsilon, \delta], U(f; t)$ is a UP-subalgebra of A if $U(f; t)$ is nonempty.*

Proof. Assume that f is a fuzzy UP-subalgebra with thresholds ε and δ of A. Let $t \in (\varepsilon, \delta]$ be such that $U(f;t) \neq \emptyset$ and let $x, y \in U(f;t)$. Then $f(x) \geq t, f(y) \geq t$, and $\delta \geq t$. Thus t is a lower bound of $\{f(x), f(y), \delta\}$. Since f is a fuzzy UP-subalgebra with thresholds ε and δ of A, we have

$$\begin{aligned}\max\{f(x \cdot y), \varepsilon\} &\geq \min\{f(x), f(y), \delta\} \\ &\geq t \\ &> \varepsilon.\end{aligned}$$

Thus $\max\{f(x \cdot y), \varepsilon\} = f(x \cdot y)$. Since $\max\{f(x \cdot y), \varepsilon\} \geq t$, we get $f(x \cdot y) \geq t$. Thus $x \cdot y \in U(f;t)$. Hence, $U(f;t)$ is a UP-subalgebra of A.

Conversely, assume that for all $t \in (\varepsilon, \delta]$, $U(f;t)$ is a UP-subalgebra of A if $U(f;t)$ is nonempty. Let $x, y \in A$. Then $f(x), f(y) \in [0,1]$. Choose $t = \min\{f(x), f(y)\}$. Then $f(x) \geq t$ and $f(y) \geq t$. Thus $x, y \in U(f;t) \neq \emptyset$. By assumption, we have $U(f;t)$ is a UP-subalgebra of A. So $x \cdot y \in U(f;t)$, that is, $f(x \cdot y) \geq t = \min\{f(x), f(y)\}$. Thus

$$\begin{aligned}\max\{f(x \cdot y), \varepsilon\} &\geq f(x \cdot y) \\ &\geq \min\{f(x), f(y)\} \\ &\geq \min\{f(x), f(y), \delta\}.\end{aligned}$$

Hence, f is a UP-subalgebra with thresholds ε and δ of A. □

Theorem 3.2.2. *Let f be a fuzzy set in A and $\varepsilon, \delta \in [0,1]$ with $\varepsilon < \delta$. Then f is a fuzzy UP-filter with thresholds ε and δ of A if and only if for all $t \in (\varepsilon, \delta], U(f;t)$ is a UP-filter of A if $U(f;t)$ is nonempty.*

Proof. Assume that f is a fuzzy UP-filter with thresholds ε and δ of A. Let $t \in (\varepsilon, \delta]$ be such that $U(f;t) \neq \emptyset$ and let $a \in U(f;t)$. Then $f(a) \geq t$ and $\delta \geq t$. Thus t is a lower bound of $\{f(a), \delta\}$. Since f is a fuzzy UP-filter with thresholds ε and δ of A, we have

$$\begin{aligned}\max\{f(0), \varepsilon\} &\geq \min\{f(a), \delta\} \\ &\geq t \\ &> \varepsilon.\end{aligned}$$

Thus $\max\{f(0),\varepsilon\} = f(0)$. Since $\max\{f(0),\varepsilon\} \geq t$, we get $f(0) \geq t$. Thus $0 \in U(f;t)$. Next, let $x, y \in A$ be such that $x \cdot y \in U(f;t)$ and $x \in U(f;t)$. Then $f(x \cdot y) \geq t, f(x) \geq t$, and $\delta \geq t$. Thus t is a lower bound of $\{f(x \cdot y), f(x), \delta\}$. Since f is a fuzzy UP-filter with thresholds ε and δ of A, we have

$$\begin{aligned}\max\{f(y),\varepsilon\} &\geq \min\{f(x \cdot y), f(x), \delta\}\\ &\geq t\\ &> \varepsilon.\end{aligned}$$

Thus $\max\{f(y),\varepsilon\} = f(y)$. Since $\max\{f(y),\varepsilon\} \geq t$, we get $f(y) \geq t$. Thus $y \in U(f;t)$. Hence, $U(f;t)$ is a UP-filter of A.

Conversely, assume that for all $t \in (\varepsilon,\delta]$, $U(f;t)$ is a UP-filter of A if $U(f;t)$ is nonempty. Let $x \in A$. Then $f(x) \in [0,1]$. Choose $t = f(x)$. Then $f(x) \geq t$. Thus $x \in U(f;t) \neq \emptyset$. By assumption, we have $U(f;t)$ is a UP-filter of A. So $0 \in U(f;t)$, that is, $f(0) \geq t = f(x)$, so

$$\begin{aligned}\max\{f(0),\varepsilon\} &\geq f(0)\\ &\geq f(x)\\ &\geq \min\{f(x),\delta\}.\end{aligned}$$

Next, let $x, y \in A$. Then $f(x \cdot y), f(x) \in [0,1]$. Choose $t = \min\{f(x \cdot y), f(x)\}$. Then $f(x \cdot y) \geq t$ and $f(x) \geq t$. Thus $x \cdot y, x \in U(f;t) \neq \emptyset$. By assumption, we have $U(f;t)$ is a UP-filter of A. So $y \in U(f;t)$, that is, $f(y) \geq t = \min\{f(x \cdot y), f(x)\}$. Thus

$$\begin{aligned}\max\{f(y),\varepsilon\} &\geq f(y)\\ &\geq \min\{f(x \cdot y), f(x)\}\\ &\geq \min\{f(x \cdot y), f(x), \delta\}.\end{aligned}$$

f is a fuzzy UP-filter with thresholds ε and δ of A. □

Theorem 3.2.3. *Let f be a fuzzy set in A and $\varepsilon, \delta \in [0,1]$ with $\varepsilon < \delta$. Then f is a fuzzy UP-ideal with thresholds ε and δ of A if and only if for all $t \in (\varepsilon,\delta], U(f;t)$ is a UP-ideal of A if $U(f;t)$ is nonempty.*

Proof. Assume that f is a fuzzy UP-ideal with thresholds ε and δ of A. Let $t \in (\varepsilon, \delta]$ be such that $U(f;t) \neq \emptyset$ and let $a \in U(f;t)$. Then $f(a) \geq t$ and $\delta \geq t$. Thus t is a lower bound of $\{f(a), \delta\}$. Since f is a fuzzy UP-ideal with thresholds ε and δ of A, we have

$$\begin{aligned}\max\{f(0), \varepsilon\} &\geq \min\{f(a), \delta\} \\ &\geq t \\ &> \varepsilon.\end{aligned}$$

Thus $\max\{f(0), \varepsilon\} = f(0)$. Since $\max\{f(0), \varepsilon\} \geq t$, we get $f(0) \geq t$. Thus $0 \in U(f;t)$. Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in U(f;t)$ and $y \in U(f;t)$. Then $f(x \cdot (y \cdot z)) \geq t$, $f(y) \geq t$, and $\delta \geq t$. Thus t is a lower bound of $\{f(x \cdot (y \cdot z)), f(y), \delta\}$. Since f is a fuzzy UP-ideal with thresholds ε and δ of A, we have

$$\begin{aligned}\max\{f(x \cdot z), \varepsilon\} &\geq \min\{f(x \cdot (y \cdot z)), f(y), \delta\} \\ &\geq t \\ &> \varepsilon.\end{aligned}$$

Thus $\max\{f(x \cdot z), \varepsilon\} = f(x \cdot z)$. Since $\max\{f(x \cdot z), \varepsilon\} \geq t$, we get $f(x \cdot z) \geq t$. Thus $x \cdot z \in U(f;t)$. Hence, $U(f;t)$ is a UP-ideal of A.

Conversely, assume that for all $t \in (\varepsilon, \delta]$, $U(f;t)$ is a UP-ideal of A if $U(f;t)$ is nonempty. Let $x \in A$. Then $f(x) \in [0, 1]$. Choose $t = f(x)$. Then $f(x) \geq t$. Thus $x \in U(f;t) \neq \emptyset$. By assumption, we have $U(f;t)$ is a UP-ideal of A. So $0 \in U(f;t)$, that is, $f(0) \geq t = f(x)$, so

$$\begin{aligned}\max\{f(0), \varepsilon\} &\geq f(0) \\ &\geq f(x) \\ &\geq \min\{f(x), \delta\}.\end{aligned}$$

Next, let $x, y, z \in A$. Then $f(x \cdot (y \cdot z)), f(y) \in [0, 1]$. Choose $t = \min\{f(x \cdot (y \cdot z)), f(y)\}$. Then $f(x \cdot (y \cdot z)) \geq t$ and $f(y) \geq t$. Thus $x \cdot (y \cdot z), y \in U(f;t) \neq \emptyset$. By assumption, we have $U(f;t)$ is a UP-ideal of A. So $x \cdot z \in U(f;t)$, that is,

$f(x \cdot z) \geq t = \min\{f(x \cdot (y \cdot z)), f(y)\}$. Thus

$$\begin{aligned}\max\{f(x \cdot z), \varepsilon\} &\geq f(x \cdot z) \\ &\geq \min\{f(x \cdot (y \cdot z)), f(y)\} \\ &\geq \min\{f(x \cdot (y \cdot z)), f(y), \delta\}.\end{aligned}$$

Hence, f is a fuzzy UP-ideal with thresholds ε and δ of A. □

Theorem 3.2.4. *Let f be a fuzzy set in A and $\varepsilon, \delta \in [0,1]$ with $\varepsilon < \delta$. Then f is a fuzzy strongly UP-ideal with thresholds ε and δ of A if and only if for all $t \in (\varepsilon, \delta], U(f;t)$ is a strongly UP-ideal of A if $U(f;t)$ is nonempty.*

Proof. Assume that f is a fuzzy strongly UP-ideal with thresholds ε and δ of A. Let $t \in (\varepsilon, \delta]$ be such that $U(f;t) \neq \emptyset$ and let $a \in U(f;t)$. Then $f(a) \geq t$ and $\delta \geq t$. Thus t is a lower bound of $\{f(a), \delta\}$. Since f is a fuzzy strongly UP-ideal with thresholds ε and δ of A, we have

$$\begin{aligned}\max\{f(0), \varepsilon\} &\geq \min\{f(a), \delta\} \\ &\geq t \\ &> \varepsilon.\end{aligned}$$

Thus $\max\{f(0), \varepsilon\} = f(0)$. Since $\max\{f(0), \varepsilon\} \geq t$, we get $f(0) \geq t$. Thus $0 \in U(f;t)$. Next, let $x, y, z \in A$ be such that $(z \cdot y) \cdot (z \cdot x) \in U(f;t)$ and $y \in U(f;t)$. Then $f((z \cdot y) \cdot (z \cdot x)) \geq t, f(y) \geq t$, and $\delta \geq t$. Thus t is a lower bound of $\{f((z \cdot y) \cdot (z \cdot x)), f(y), \delta\}$. Since f is a fuzzy strongly UP-ideal with thresholds ε and δ of A, we have

$$\begin{aligned}\max\{f(x), \varepsilon\} &\geq \min\{f((z \cdot y) \cdot (z \cdot x)), f(y), \delta\} \\ &\geq t \\ &> \varepsilon.\end{aligned}$$

Thus $\max\{f(x), \varepsilon\} = f(x)$. Since $\max\{f(x), \varepsilon\} \geq t$, we get $f(x) \geq t$. Thus $x \in U(f;t)$. Hence, $U(f;t)$ is a strongly UP-ideal of A.

Conversely, assume that for all $t \in (\varepsilon, \delta]$, $U(f;t)$ is a strongly UP-ideal of A if $U(f;t)$ is nonempty. Let $x \in A$. Then $f(x) \in [0,1]$. Choose $t = f(x)$. Then

$f(x) \geq t$. Thus $x \in U(f;t) \neq \emptyset$. By assumption, we have $U(f;t)$ is a strongly UP-ideal of A. So $0 \in U(f;t)$, that is, $f(0) \geq t = f(x)$, so

$$\begin{aligned}\max\{f(0), \varepsilon\} &\geq f(0)\\ &\geq f(x)\\ &\geq \min\{f(x), \delta\}.\end{aligned}$$

Next, let $x, y, z \in A$. Then $f((z\cdot y)\cdot(z\cdot x)), f(y) \in [0,1]$. Choose $t = \min\{f((z\cdot y)\cdot(z\cdot x)), f(y)\}$. Then $f((z\cdot y)\cdot(z\cdot x)) \geq t$ and $f(y) \geq t$. Thus $(z\cdot y)\cdot(z\cdot x), y \in U(f;t) \neq \emptyset$. By assumption, we have $U(f;t)$ is a strongly UP-ideal of A. So $x \in U(f;t)$, that is, $f(x) \geq t = \min\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$. Thus

$$\begin{aligned}\max\{f(x), \varepsilon\} &\geq f(x)\\ &\geq \min\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}\\ &\geq \min\{f((z \cdot y) \cdot (z \cdot x)), f(y), \delta\}.\end{aligned}$$

Hence, f is a fuzzy strongly UP-ideal with thresholds ε and δ of A. □

3.3 Image and Preimage of a Fuzzy Set

Definition 3.3.1. [3] Let f be a function from a nonempty set X to a nonempty set Y. If μ is a fuzzy set in X, then the fuzzy set β in Y defined by

$$\beta(y) = \begin{cases} \sup\{\mu(t)\}_{t\in f^{-1}(y)} & \text{if } f^{-1}(y) \neq \emptyset,\\ 0 & \text{if otherwise}\end{cases}$$

is said to be the *image of μ under f*. Similarly, if β is a fuzzy set in Y, then the fuzzy set $\mu = \beta \circ f$ in X (i.e., the fuzzy set defined by $\mu(x) = \beta(f(x))$ for all $x \in X$) is called the *preimage of β under f*.

Definition 3.3.2. [10] A fuzzy set f in A has sup *property* if for any nonempty subset T of A, there exists $t_0 \in T$ such that $f(t_0) = \sup\{f(t)\}_{t\in T}$.

Lemma 3.3.3. *[9] Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras and let $f\colon A \to B$ be a UP-epimorphism. Let μ be an f-invariant fuzzy set in A with* sup *property.*

For any $a, b \in B$, there exist $a_0 \in f^{-1}(a)$ and $b_0 \in f^{-1}(b)$ such that $\beta(a) = \mu(a_0), \beta(b) = \mu(b_0)$, and $\beta(a * b) = \mu(a_0 \cdot b_0)$.

Theorem 3.3.4. *Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras and let $f\colon A \to B$ be a UP-epimorphism. Then the following statements hold:*

(1) if μ is an f-invariant fuzzy UP-subalgebra with thresholds ε and δ of A with sup *property, then β is a fuzzy UP-subalgebra with thresholds ε and δ of B,*

(2) if μ is an f-invariant fuzzy UP-filter with thresholds ε and δ of A with sup *property, then β is a fuzzy UP-filter with thresholds ε and δ of B,*

(3) if μ is an f-invariant fuzzy UP-ideal with thresholds ε and δ of A with sup *property, then β is a fuzzy UP-ideal with thresholds ε and δ of B, and*

(4) if μ is an f-invariant fuzzy strongly UP-ideal with thresholds ε and δ of A with sup *property, then β is a fuzzy strongly UP-ideal with thresholds ε and δ of B.*

Proof. (1) Assume that μ is an f-invariant fuzzy UP-subalgebra with thresholds ε and δ of A with sup property. Let $a, b \in B$. By Lemma 3.3.3, there exist $a_0 \in f^{-1}(a)$ and $b_0 \in f^{-1}(b)$ such that $\beta(a) = \mu(a_0), \beta(b) = \mu(b_0)$, and $\beta(a * b) = \mu(a_0 \cdot b_0)$. Thus

$$\begin{aligned}\max\{\beta(a * b), \varepsilon\} &= \max\{\mu(a_0 \cdot b_0), \varepsilon\}\\ &\geq \min\{\mu(a_0), \mu(b_0), \delta\}\\ &= \min\{\beta(a), \beta(b), \delta\}.\end{aligned}$$

Hence, β is a fuzzy UP-subalgebra with thresholds ε and δ of B.

(2) Assume that μ is an f-invariant fuzzy UP-filter with thresholds ε and δ of A with sup property. Since $f(0_A) = 0_B$, we have $f^{-1}(0_B) \neq \emptyset$. Then there exists $x_1 \in f^{-1}(0_B)$ such that $\mu(x_1) = \beta(0_B)$. Thus $f(x_1) = 0_B = f(0_A)$, so

$\mu(x_1) = \mu(0_A)$ because μ is f-invariant. Hence, $\mu(0_A) = \beta(0_B)$. Let $y \in B$. Since f is surjective, we have $f^{-1}(y) \neq \emptyset$. Then there exists $x \in f^{-1}(y)$ such that $\mu(x) = \beta(y)$, so

$$\begin{aligned}\max\{\beta(0_B), \varepsilon\} &= \max\{\mu(0_A), \varepsilon\} \\ &\geq \min\{\mu(x), \delta\} \\ &= \min\{\beta(y), \delta\}.\end{aligned}$$

Next, let $a, b \in B$. By Lemma 3.3.3, there exist $a_0 \in f^{-1}(a)$ and $b_0 \in f^{-1}(b)$ such that $\mu(a_0) = \beta(a), \mu(b_0) = \beta(b)$, and $\mu(a_0 \cdot b_0) = \beta(a * b)$. Thus

$$\begin{aligned}\max\{\beta(b), \varepsilon\} &= \max\{\mu(b_0), \varepsilon\} \\ &\geq \min\{\mu(a_0 \cdot b_0), \mu(a_0), \delta\} \\ &= \min\{\beta(a * b), \beta(a), \delta\}.\end{aligned}$$

Hence, β is a fuzzy UP-filter with thresholds ε and δ of B.

(3) Assume that μ is an f-invariant fuzzy UP-ideal with thresholds ε and δ of A with sup property. Since $f(0_A) = 0_B$, we have $f^{-1}(0_B) \neq \emptyset$. Then there exists $x_1 \in f^{-1}(0_B)$ such that $\mu(x_1) = \beta(0_B)$. Thus $f(x_1) = 0_B = f(0_A)$, so $\mu(x_1) = \mu(0_A)$ because μ is f-invariant. Hence, $\mu(0_A) = \beta(0_B)$. Let $y \in B$. Since f is surjective, we have $f^{-1}(y) \neq \emptyset$. Then there exists $x \in f^{-1}(y)$ such that $\mu(x) = \beta(y)$, so

$$\begin{aligned}\max\{\beta(0_B), \varepsilon\} &= \max\{\mu(0_A), \varepsilon\} \\ &\geq \min\{\mu(x), \delta\} \\ &= \min\{\beta(y), \delta\}.\end{aligned}$$

Next, let $a, b, c \in B$. By Lemma 3.3.3, there exist $a_0 \in f^{-1}(a), b_0 \in f^{-1}(b)$, and $c_0 \in f^{-1}(c)$ such that $\beta(b) = \mu(b_0), \beta(a * c) = \mu(a_0 \cdot c_0)$, and $\beta(a * (b * c)) = \mu(a_0 \cdot (b_0 \cdot c_0))$. Thus

$$\begin{aligned}\max\{\beta(a * c), \varepsilon\} &= \max\{\mu(a_0 \cdot c_0), \varepsilon\} \\ &\geq \min\{\mu(a_0 \cdot (b_0 \cdot c_0)), \mu(b_0), \delta\} \\ &= \min\{\beta(a * (b * c)), \beta(b), \delta\}.\end{aligned}$$

Hence, β is a fuzzy UP-ideal with thresholds ε and δ of B.

(4) Assume that μ is an f-invariant fuzzy strongly UP-ideal with thresholds ε and δ of A with sup property. Since $f(0_A) = 0_B$, we have $f^{-1}(0_B) \neq \emptyset$. Then there exists $x_1 \in f^{-1}(0_B)$ such that $\mu(x_1) = \beta(0_B)$. Thus $f(x_1) = 0_B = f(0_A)$, so $\mu(x_1) = \mu(0_A)$ because μ is f-invariant. Hence, $\mu(0_A) = \beta(0_B)$. Let $y \in B$. Since f is surjective, we have $f^{-1}(y) \neq \emptyset$. Then there exists $x \in f^{-1}(y)$ such that $\mu(x) = \beta(y)$, so

$$\begin{aligned}\max\{\beta(0_B), \varepsilon\} &= \max\{\mu(0_A), \varepsilon\} \\ &\geq \min\{\mu(x), \delta\} \\ &= \min\{\beta(y), \delta\}.\end{aligned}$$

Next, let $a, b, c \in B$. By Lemma 3.3.3, there exist $a_0 \in f^{-1}(a), b_0 \in f^{-1}(b)$, and $c_0 \in f^{-1}(c)$ such that $\mu(a_0) = \beta(a), \mu(b_0) = \beta(b)$, and $\mu((c_0 \cdot b_0) \cdot (c_0 \cdot a_0)) = \beta((c \cdot b) \cdot (c \cdot a))$. Thus

$$\begin{aligned}\max\{\beta(a), \varepsilon\} &= \max\{\mu(a_0), \varepsilon\} \\ &\geq \min\{\mu((c_0 \cdot b_0) \cdot (c_0 \cdot a_0)), \mu(b_0), \delta\} \\ &= \min\{\beta((c \cdot b) \cdot (c \cdot a)), \beta(b), \delta\}.\end{aligned}$$

Hence, β is a fuzzy strongly UP-ideal with thresholds ε and δ of B. □

Theorem 3.3.5. *Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras and let $f \colon A \to B$ be a UP-homomorphism. Then the following statements hold:*

(1) if β is a fuzzy UP-subalgebra with thresholds ε and δ of B, then μ is a fuzzy UP-subalgebra with thresholds ε and δ of A,

(2) if β is a fuzzy UP-filter with thresholds ε and δ of B, then μ is a fuzzy UP-filter with thresholds ε and δ of A,

(3) if β is a fuzzy UP-ideal with thresholds ε and δ of B, then μ is a fuzzy UP-ideal with thresholds ε and δ of A, and

(4) if β is a fuzzy strongly UP-ideal with thresholds ε and δ of B, then μ is a fuzzy strongly UP-ideal with thresholds ε and δ of A.

Proof. (1) Assume that β is a fuzzy UP-subalgebra with thresholds ε and δ of B. Let $x, y \in A$. Then

$$\begin{aligned}
\max\{\mu(x \cdot y), \varepsilon\} &= \max\{(\beta \circ f)(x \cdot y), \varepsilon\} \\
&= \max\{\beta(f(x \cdot y)), \varepsilon\} \\
&= \max\{\beta(f(x) * f(y)), \varepsilon\} \\
&\geq \min\{\beta(f(x)), \beta(f(y)), \delta\} \\
&= \min\{(\beta \circ f)(x), (\beta \circ f)(y), \delta\} \\
&= \min\{\mu(x), \mu(y), \delta\}.
\end{aligned}$$

Hence, μ is a fuzzy UP-subalgebra with thresholds ε and δ of A.

(2) Assume that β is a fuzzy UP-filter with thresholds ε and δ of B. Let $x \in A$. Then

$$\begin{aligned}
\max\{\mu(0_A), \varepsilon\} &= \max\{(\beta \circ f)(0_A), \varepsilon\} \\
&= \max\{\beta(f(0_A)), \varepsilon\} \\
&= \max\{\beta(0_B), \varepsilon\} && (f(0_A) = 0_B) \\
&\geq \min\{\beta(f(x)), \delta\} \\
&= \min\{(\beta \circ f)(x), \delta\} \\
&= \min\{\mu(x), \delta\}.
\end{aligned}$$

Let $x, y \in A$. Then

$$\begin{aligned}
\max\{\mu(y), \varepsilon\} &= \max\{(\beta \circ f)(y), \varepsilon\} \\
&= \max\{\beta(f(y)), \varepsilon\} \\
&\geq \min\{\beta(f(x) * f(y)), \beta(f(x)), \delta\} \\
&= \min\{\beta(f(x \cdot y)), \beta(f(x)), \delta\} \\
&= \min\{(\beta \circ f)(x \cdot y), (\beta \circ f)(x), \delta\} \\
&= \min\{\mu(x \cdot y), \mu(x), \delta\}.
\end{aligned}$$

Hence, μ is a fuzzy UP-filter with thresholds ε and δ of A.

(3) Assume that β is a fuzzy UP-ideal with thresholds ε and δ of B. Let $x \in A$. Then

$$\begin{aligned}
\max\{\mu(0_A), \varepsilon\} &= \max\{(\beta \circ f)(0_A), \varepsilon\} \\
&= \max\{\beta(f(0_A)), \varepsilon\} \\
&= \max\{\beta(0_B), \varepsilon\} && (f(0_A) = 0_B) \\
&\geq \min\{\beta(f(x)), \delta\} \\
&= \min\{(\beta \circ f)(x), \delta\} \\
&= \min\{\mu(x), \delta\}.
\end{aligned}$$

Let $x, y, z \in A$. Then

$$\begin{aligned}
\max\{\mu(x \cdot z), \varepsilon\} &= \max\{(\beta \circ f)(x \cdot z), \varepsilon\} \\
&= \max\{\beta(f(x \cdot z)), \varepsilon\} \\
&= \min\{\beta(f(x) * f(z)), \delta\} \\
&\geq \min\{\beta(f(x) * (f(y) * f(z))), \beta(f(y)), \delta\} \\
&= \min\{\beta(f(x) * f(y \cdot z)), \beta(f(y)), \delta\} \\
&= \min\{\beta(f(x \cdot (y \cdot z))), \beta(f(y)), \delta\} \\
&= \min\{(\beta \circ f)(x \cdot (y \cdot z)), (\beta \circ f)(y), \delta\} \\
&= \min\{\mu(x \cdot (y \cdot z)), \mu(y), \delta\}.
\end{aligned}$$

Hence, μ is a fuzzy UP-ideal with thresholds ε and δ of A.

(4) Assume that β is a fuzzy strongly UP-ideal with thresholds ε and δ of B. Let $x \in A$. Then

$$\begin{aligned}
\max\{\mu(0_A), \varepsilon\} &= \max\{(\beta \circ f)(0_A), \varepsilon\} \\
&= \max\{\beta(f(0_A)), \varepsilon\} \\
&= \max\{\beta(0_B), \varepsilon\} && (f(0_A) = 0_B) \\
&\geq \min\{\beta(f(x)), \delta\} \\
&= \min\{(\beta \circ f)(x), \delta\} \\
&= \min\{\mu(x), \delta\}.
\end{aligned}$$

Let $x, y, z \in A$. Then

$$\begin{aligned}
\max\{\mu(x), \varepsilon\} &= \max\{(\beta \circ f)(x), \varepsilon\} \\
&= \max\{\beta(f(x)), \varepsilon\} \\
&\geq \min\{\beta((f(z) * f(y)) * (f(z) * f(x))), \beta(f(y)), \delta\} \\
&= \min\{\beta(f(z \cdot y) * f(z \cdot x)), \beta(f(y)), \delta\} \\
&= \min\{\beta(f((z \cdot y) \cdot (z \cdot x)), \beta(f(y)), \delta\} \\
&= \min\{(\beta \circ f)((z \cdot y) \cdot (z \cdot x)), (\beta \circ f)(y), \delta\} \\
&= \min\{\mu((z \cdot y) \cdot (z \cdot x)), \mu(y), \delta\}.
\end{aligned}$$

Hence, μ is a fuzzy strongly UP-ideal with thresholds ε and δ of A. □

Definition 3.3.6. Let f be a function from a nonempty set X to a nonempty set Y. If μ is a fuzzy set in X, then the fuzzy set β in Y defined by

$$\beta(y) = \begin{cases} \inf\{\mu(t)\}_{t \in f^{-1}(y)} & \text{if } f^{-1}(y) \neq \emptyset, \\ \\ 1 & \text{if otherwise} \end{cases}$$

is said to be the *image of μ under f*. Similarly, if β is a fuzzy set in Y, then the fuzzy set $\mu = \beta \circ f$ in X (i.e., the fuzzy set defined by $\mu(x) = \beta(f(x))$ for all $x \in X$) is called the *preimage of β under f*.

Definition 3.3.7. [10] A fuzzy set f in A has inf *property* if for any nonempty subset T of A, there exists $t_0 \in T$ such that $f(t_0) = \inf\{f(t)\}_{t \in T}$.

Lemma 3.3.8. *[6] Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras and let $f \colon A \to B$ be a UP-epimorphism. Let μ be an f-invariant fuzzy set in A with* inf *property. For any $a, b \in B$, there exist $a_0 \in f^{-1}(a)$ and $b_0 \in f^{-1}(b)$ such that $\beta(a) = \mu(a_0), \beta(b) = \mu(b_0)$, and $\beta(a * b) = \mu(a_0 \cdot b_0)$.*

Theorem 3.3.9. *Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras and let $f \colon A \to B$ be a UP-epimorphism. Then the following statements hold:*

(1) if μ is an f-invariant fuzzy UP-subalgebra with thresholds ε and δ of A with inf *property, then β is a fuzzy UP-subalgebra with thresholds ε and δ of B,*

(2) if μ is an f-invariant fuzzy UP-filter with thresholds ε and δ of A with inf *property, then β is a fuzzy UP-filter with thresholds ε and δ of B,*

(3) if μ is an f-invariant fuzzy UP-ideal with thresholds ε and δ of A with inf *property, then β is a fuzzy UP-ideal with thresholds ε and δ of B, and*

(4) if μ is an f-invariant fuzzy strongly UP-ideal with thresholds ε and δ of A with inf *property, then β is a fuzzy strongly UP-ideal with thresholds ε and δ of B.*

Proof. (1) Assume that μ is an f-invariant fuzzy UP-subalgebra with thresholds ε and δ of A with inf property. Let $a, b \in B$. By Lemma 3.3.8, there exist $a_0 \in f^{-1}(a)$ and $b_0 \in f^{-1}(b)$ such that $\beta(a) = \mu(a_0), \beta(b) = \mu(b_0)$, and $\beta(a * b) = \mu(a_0 \cdot b_0)$. Thus

$$\begin{aligned}\max\{\beta(a * b), \varepsilon\} &= \max\{\mu(a_0 \cdot b_0), \varepsilon\}\\ &\geq \min\{\mu(a_0), \mu(b_0), \delta\}\\ &= \min\{\beta(a), \beta(b), \delta\}.\end{aligned}$$

Hence, β is a fuzzy UP-subalgebra with thresholds ε and δ of B.

(2) Assume that μ is an f-invariant fuzzy UP-filter with thresholds ε and δ of A with inf property. Since $f(0_A) = 0_B$, we have $f^{-1}(0_B) \neq \emptyset$. Then there exist $x_1 \in f^{-1}(0_B)$ such that $\mu(x_1) = \beta(0_B)$. Thus $f(x_1) = 0_B = f(0_A)$, so $\mu(x_1) = \mu(0_A)$ because μ is f-invariant. Hence, $\mu(0_A) = \beta(0_B)$. Let $y \in B$. Since f is surjective, we have $f^{-1}(y) \neq \emptyset$. Then there exist $x \in f^{-1}(y)$ such that $\mu(x) = \beta(y)$, so

$$\begin{aligned}\max\{\beta(0_B), \varepsilon\} &= \max\{\mu(0_A), \varepsilon\}\\ &\geq \min\{\mu(x), \delta\}\\ &= \min\{\beta(y), \delta\}.\end{aligned}$$

Next, let $a, b \in B$. By Lemma 3.3.8, there exist $a_0 \in f^{-1}(a)$ and $b_0 \in f^{-1}(b)$ such that $\beta(a) = \mu(a_0), \beta(b) = \mu(b_0)$, and $\beta(a * b) = \mu(a_0 \cdot b_0)$. Thus

$$\begin{aligned}\max\{\beta(b), \varepsilon\} &= \max\{\mu(b_0), \varepsilon\} \\ &\geq \min\{\mu(a_0 \cdot b_0), \mu(a_0), \delta\} \\ &= \min\{\beta(a * b), \beta(a), \delta\}.\end{aligned}$$

Hence, β is a fuzzy UP-filter with thresholds ε and δ of B.

(3) Assume that μ is an f-invariant fuzzy UP-ideal with thresholds ε and δ of A with inf property. Since $f(0_A) = 0_B$, we have $f^{-1}(0_B) \neq \emptyset$. Then there exist $x_1 \in f^{-1}(0_B)$ such that $\mu(x_1) = \beta(0_B)$. Thus $f(x_1) = 0_B = f(0_A)$, so $\mu(x_1) = \mu(0_A)$ because μ is f-invariant. Hence, $\mu(0_A) = \beta(0_B)$. Let $y \in B$. Since f is surjective, we have $f^{-1}(y) \neq \emptyset$. Then there exist $x \in f^{-1}(y)$ such that $\mu(x) = \beta(y)$, so

$$\begin{aligned}\max\{\beta(0_B), \varepsilon\} &= \max\{\mu(0_A), \varepsilon\} \\ &\geq \min\{\mu(x), \delta\} \\ &= \min\{\beta(y), \delta\}.\end{aligned}$$

Next, let $a, b, c \in B$. By Lemma 3.3.8, there exist $a_0 \in f^{-1}(a), b_0 \in f^{-1}(b)$, and $c_0 \in f^{-1}(c)$ such that $\beta(b) = \mu(b_0), \beta(a * c) = \mu(a_0 \cdot c_0)$, and $\beta(a * (b * c)) = \mu(a_0 \cdot (b_0 \cdot c_0))$. Thus

$$\begin{aligned}\max\{\beta(a * c), \varepsilon\} &= \max\{\mu(a_0 \cdot c_0), \varepsilon\} \\ &\geq \min\{\mu(a_0 \cdot (b_0 \cdot c_0)), \mu(b_0), \delta\} \\ &= \min\{\beta(a * (b * c)), \beta(b), \delta\}.\end{aligned}$$

Hence, β is a fuzzy UP-ideal with thresholds ε and δ of B.

(4) Assume that μ is an f-invariant fuzzy strongly UP-ideal with thresholds ε and δ of A with inf property. Since $f(0_A) = 0_B$, we have $f^{-1}(0_B) \neq \emptyset$. Then there exist $x_1 \in f^{-1}(0_B)$ such that $\mu(x_1) = \beta(0_B)$. Thus $f(x_1) = 0_B = f(0_A)$, so $\mu(x_1) = \mu(0_A)$ because μ is f-invariant. Hence, $\mu(0_A) = \beta(0_B)$. Let $y \in B$. Since f is surjective, we have $f^{-1}(y) \neq \emptyset$. Then there exist $x \in f^{-1}(y)$ such that

$\mu(x) = \beta(y)$, so

$$\begin{aligned}\max\{\beta(0_B), \varepsilon\} &= \max\{\mu(0_A), \varepsilon\}\\ &\geq \min\{\mu(x), \delta\}\\ &= \min\{\beta(y), \delta\}.\end{aligned}$$

Next, let $a, b, c \in B$. By Lemma 3.3.8, there exist $a_0 \in f^{-1}(a), b_0 \in f^{-1}(b)$, and $c_0 \in f^{-1}(c)$ such that $\beta(a) = \mu(a_0), \beta(b) = \mu(b_0)$, and $\beta((c * b) * (c * a)) = \mu((c_0 \cdot b_0) \cdot (c_0 \cdot a_0))$. Thus

$$\begin{aligned}\max\{\beta(a), \varepsilon\} &= \max\{\mu(a_0), \varepsilon\}\\ &\geq \min\{\mu((c_0 \cdot b_0) \cdot (c_0 \cdot a_0)), \mu(b_0), \delta\}\\ &= \min\{\beta((c * b) * (c * a)), \beta(b), \delta\}.\end{aligned}$$

Hence, β is a fuzzy strongly UP-ideal with thresholds ε and δ of B. □

Theorem 3.3.10. *Let $(A; \cdot, 0_A)$ and $(B; *, 0_B)$ be UP-algebras and let $f: A \to B$ be a UP-homomorphism. Then the following statements hold:*

(1) if β is a fuzzy UP-subalgebra with thresholds ε and δ of B, then μ is a fuzzy UP-subalgebra with thresholds ε and δ of A,

(2) if β is a fuzzy UP-filter with thresholds ε and δ of B, then μ is a fuzzy UP-filter with thresholds ε and δ of A,

(3) if β is a fuzzy UP-ideal with thresholds ε and δ of B, then μ is a fuzzy UP-ideal with thresholds ε and δ of A, and

(4) if β is a fuzzy strongly UP-ideal with thresholds ε and δ of B, then μ is a fuzzy strongly UP-ideal with thresholds ε and δ of A.

Proof. (1) Assume that β is a fuzzy UP-subalgebra with thresholds ε and δ of B.

Let $x, y \in A$. Then

$$\begin{aligned}
\max\{\mu(x \cdot y), \varepsilon\} &= \max\{(\beta \circ f)(x \cdot y), \varepsilon\} \\
&= \max\{\beta(f(x \cdot y)), \varepsilon\} \\
&= \max\{\beta(f(x) * f(y)), \varepsilon\} \\
&\geq \min\{\beta(f(x)), \beta(f(y)), \delta\} \\
&= \min\{(\beta \circ f)(x), (\beta \circ f)(y), \delta\} \\
&= \min\{\mu(x), \mu(y), \delta\}.
\end{aligned}$$

Hence, μ is a fuzzy UP-subalgebra with thresholds ε and δ of A.

(2) Assume that β is a fuzzy UP-filer with thresholds ε and δ of B. Let $x \in A$. Then

$$\begin{aligned}
\max\{\mu(0_A), \varepsilon\} &= \max\{(\beta \circ f)(0_A), \varepsilon\} \\
&= \max\{\beta(f(0_A)), \varepsilon\} \\
&= \max\{\beta(0_B), \varepsilon\} && (f(0_A) = 0_B) \\
&\geq \min\{\beta(f(x)), \delta\} \\
&= \min\{(\beta \circ f)(x), \delta\} \\
&= \min\{\mu(x), \delta\}.
\end{aligned}$$

Let $x, y \in A$. Then

$$\begin{aligned}
\max\{\mu(y), \varepsilon\} &= \max\{(\beta \circ f)(y), \varepsilon\} \\
&= \max\{\beta(f(y)), \varepsilon\} \\
&\geq \min\{\beta(f(x) * f(y)), \beta(f(x)), \delta\} \\
&= \min\{\beta(f(x \cdot y)), \beta(f(x)), \delta\} \\
&= \min\{(\beta \circ f)(x \cdot y), (\beta \circ f)(x), \delta\} \\
&= \min\{\mu(x \cdot y), \mu(x), \delta\}.
\end{aligned}$$

Hence, μ is a fuzzy UP-filter with thresholds ε and δ of A.

(3) Assume that β is a fuzzy UP-ideal with thresholds ε and δ of B. Let

$x \in A$. Then

$$\begin{aligned}
\max\{\mu(0_A), \varepsilon\} &= \max\{(\beta \circ f)(0_A), \varepsilon\} \\
&= \max\{\beta(f(0_A)), \varepsilon\} \\
&= \max\{\beta(0_B), \varepsilon\} && (f(0_A) = 0_B) \\
&\geq \min\{\beta(f(x)), \delta\} \\
&= \min\{(\beta \circ f)(x), \delta\} \\
&= \min\{\mu(x), \delta\}.
\end{aligned}$$

Let $x, y, z \in A$. Then

$$\begin{aligned}
\max\{\mu(x \cdot z), \varepsilon\} &= \max\{(\beta \circ f)(x \cdot z), \varepsilon\} \\
&= \max\{\beta(f(x \cdot z)), \varepsilon\} \\
&= \max\{\beta(f(x) * f(z)), \varepsilon\} \\
&\geq \min\{\beta(f(x) * (f(y) * f(z))), \beta(f(y)), \delta\} \\
&= \min\{\beta(f(x) * f(y \cdot z)), \beta(f(y)), \delta\} \\
&= \min\{\beta(f(x \cdot (y \cdot z))), \beta(f(y)), \delta\} \\
&= \min\{(\beta \circ f)(x \cdot (y \cdot z)), (\beta \circ f)(y), \delta\} \\
&= \min\{\mu(x \cdot (y \cdot z)), \mu(y), \delta\}.
\end{aligned}$$

Hence, μ is a fuzzy UP-ideal with thresholds ε and δ of A.

(4) Assume that β is a fuzzy strongly UP-ideal with thresholds ε and δ of B. Let $x \in A$. Then

$$\begin{aligned}
\max\{\mu(0_A), \varepsilon\} &= \max\{(\beta \circ f)(0_A), \varepsilon\} \\
&= \max\{\beta(f(0_A)), \varepsilon\} \\
&= \max\{\beta(0_B), \varepsilon\} && (f(0_A) = 0_B) \\
&\geq \min\{\beta(f(x)), \delta\} \\
&= \min\{(\beta \circ f)(x), \delta\} \\
&= \min\{\mu(x), \delta\}.
\end{aligned}$$

Let $x, y, z \in A$. Then

$$\begin{aligned}\max\{\mu(x), \varepsilon\} &= \max\{(\beta \circ f)(x), \varepsilon\} \\ &= \max\{\beta(f(x)), \varepsilon\} \\ &\geq \min\{\beta((f(z) * f(y)) * (f(z) * f(x))), \beta(f(y)), \delta\} \\ &= \min\{\beta(f(z \cdot y) * f(z \cdot x)), \beta(f(y)), \delta\} \\ &= \min\{\beta(f((z \cdot y) \cdot (z \cdot x))), \beta(f(y)), \delta\} \\ &= \min\{(\beta \circ f)((z \cdot y) \cdot (z \cdot x)), (\beta \circ f)(y), \delta\} \\ &= \min\{\mu((z \cdot y) \cdot (z \cdot x)), \mu(y), \delta\}.\end{aligned}$$

Hence, μ is a fuzzy strongly UP-ideal with thresholds ε and δ of A. $\square$

CHAPTER 4
Conclusions

From the study, we get the main results as the following:

1. Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras and let $f\colon A \to B$ be a UP-homomorphism. Then the following statements hold:

   (1) $f(0_A) = 0_B$, and

   (2) for any $x, y \in A$, if $x \leq y$, then $f(x) \leq f(y)$.

2. If f is a fuzzy UP-subalgebra with thresholds ε and δ of A, then $\max\{f(0), \varepsilon\} \geq \min\{f(x), \delta\}$ for all $x \in A$. In particular, if there exists $y \in A$ such that $f(y) > \delta$, then $f(0) > \varepsilon$.

3. Every fuzzy strongly UP-ideal with thresholds ε and δ of A is a fuzzy UP-ideal with thresholds ε and δ.

4. Every fuzzy UP-ideal with thresholds ε and δ of A is a fuzzy UP-filter with thresholds ε and δ.

5. Every fuzzy UP-filter with thresholds ε and δ of A is a fuzzy UP-subalgebra with thresholds ε and δ.

6. Let $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. If a fuzzy set f in A is constant, then it is a fuzzy strongly UP-ideal with thresholds ε and δ of A.

7. Let $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. If a fuzzy set f in A is such that $f(x) \leq \varepsilon$ for all $x \in A$, then it is a fuzzy strongly UP-ideal (resp. fuzzy UP-ideal, fuzzy UP-filter and fuzzy UP-subalgebra) with thresholds ε and δ of A.

8. Let $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. If a fuzzy set f in A with $\varepsilon \leq f(x) \leq \delta$ is a fuzzy strongly UP-ideal with thresholds ε and δ of A, then it is constant.

9. Let $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. If a fuzzy set f in A is such that $f(x) \geq \delta$ for all $x \in A$, then it is a fuzzy strongly UP-ideal (resp. fuzzy UP-ideal, fuzzy UP-filter and fuzzy UP-subalgebra) with thresholds ε and δ of A.

10. Let $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. Then a fuzzy set f in A with $\varepsilon \leq f(x) \leq \delta$ is a fuzzy strongly UP-ideal with thresholds ε and δ of A if and only if it is constant.

11. Let f be a fuzzy set in A and $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. Then f is a fuzzy UP-subalgebra with thresholds ε and δ of A if and only if for all $t \in (\varepsilon, \delta], U(f; t)$ is a UP-subalgebra of A if $U(f; t)$ is nonempty.

12. Let f be a fuzzy set in A and $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. Then f is a fuzzy UP-filter with thresholds ε and δ of A if and only if for all $t \in (\varepsilon, \delta], U(f; t)$ is a UP-filter of A if $U(f; t)$ is nonempty.

13. Let f be a fuzzy set in A and $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. Then f is a fuzzy UP-ideal with thresholds ε and δ of A if and only if for all $t \in (\varepsilon, \delta], U(f; t)$ is a UP-ideal of A if $U(f; t)$ is nonempty.

14. Let f be a fuzzy set in A and $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. Then f is a fuzzy strongly UP-ideal with thresholds ε and δ of A if and only if for all $t \in (\varepsilon, \delta], U(f; t)$ is a strongly UP-ideal of A if $U(f; t)$ is nonempty.

15. Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras and let $f \colon A \to B$ be a UP-epimorphism. Let μ be an f-invariant fuzzy set in A with sup property. For any $a, b \in B$, there exist $a_0 \in f^{-1}(a)$ and $b_0 \in f^{-1}(b)$ such that $\beta(a) = \mu(a_0), \beta(b) = \mu(b_0)$, and $\beta(a * b) = \mu(a_0 \cdot b_0)$.

16. Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras and let $f \colon A \to B$ be a UP-epimorphism. Then the following statements hold:

    (1) if μ is an f-invariant fuzzy UP-subalgebra with thresholds ε and δ of A with sup property, then β is a fuzzy UP-subalgebra with thresholds ε and δ of B,

    (2) if μ is an f-invariant fuzzy UP-filter with thresholds ε and δ of A with sup property, then β is a fuzzy UP-filter with thresholds ε and δ of B,

(3) if μ is an f-invariant fuzzy UP-ideal with thresholds ε and δ of A with sup property, then β is a fuzzy UP-ideal with thresholds ε and δ of B, and

(4) if μ is an f-invariant fuzzy strongly UP-ideal with thresholds ε and δ of A with sup property, then β is a fuzzy strongly UP-ideal with thresholds ε and δ of B.

17. Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras and let $f\colon A \to B$ be a UP-homomorphism. Then the following statements hold:

(1) if β is a fuzzy UP-subalgebra with thresholds ε and δ of B, then μ is a fuzzy UP-subalgebra with thresholds ε and δ of A,

(2) if β is a fuzzy UP-filter with thresholds ε and δ of B, then μ is a fuzzy UP-filter with thresholds ε and δ of A,

(3) if β is a fuzzy UP-ideal with thresholds ε and δ of B, then μ is a fuzzy UP-ideal with thresholds ε and δ of A, and

(4) if β is a fuzzy strongly UP-ideal with thresholds ε and δ of B, then μ is a fuzzy strongly UP-ideal with thresholds ε and δ of A.

18. Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras and let $f\colon A \to B$ be a UP-epimorphism. Let μ be an f-invariant fuzzy set in A with inf property. For any $a, b \in B$, there exist $a_0 \in f^{-1}(a)$ and $b_0 \in f^{-1}(b)$ such that $\beta(a) = \mu(a_0), \beta(b) = \mu(b_0)$, and $\beta(a * b) = \mu(a_0 \cdot b_0)$.

19. Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras and let $f\colon A \to B$ be a UP-epimorphism. Then the following statements hold:

(1) if μ is an f-invariant fuzzy UP-subalgebra with thresholds ε and δ of A with inf property, then β is a fuzzy UP-subalgebra with thresholds ε and δ of B,

(2) if μ is an f-invariant fuzzy UP-filter with thresholds ε and δ of A with inf property, then β is a fuzzy UP-filter with thresholds ε and δ of B,

(3) if μ is an f-invariant fuzzy UP-ideal with thresholds ε and δ of A with inf property, then β is a fuzzy UP-ideal with thresholds ε and δ of B, and

(4) if μ is an f-invariant fuzzy strongly UP-ideal with thresholds ε and δ of A with inf property, then β is a fuzzy strongly UP-ideal with thresholds ε and δ of B.

20. Let $(A; \cdot, 0_A)$ and $(B; *, 0_B)$ be UP-algebras and let $f \colon A \to B$ be a UP-homomorphism. Then the following statements hold:

(1) if β is a fuzzy UP-subalgebra with thresholds ε and δ of B, then μ is a fuzzy UP-subalgebra with thresholds ε and δ of A,

(2) if β is a fuzzy UP-filter with thresholds ε and δ of B, then μ is a fuzzy UP-filter with thresholds ε and δ of A,

(3) if β is a fuzzy UP-ideal with thresholds ε and δ of B, then μ is a fuzzy UP-ideal with thresholds ε and δ of A, and

(4) if β is a fuzzy strongly UP-ideal with thresholds ε and δ of B, then μ is a fuzzy strongly UP-ideal with thresholds ε and δ of A.

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APPENDIX

Writing for Publication: Generalized Fuzzy Sets in UP-Algebras*

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Abstract

Based on the theory of fuzzy sets, the notion of fuzzy UP-subalgebras (resp., fuzzy UP-filters, fuzzy UP-ideals, fuzzy strongly UP-ideals) with thresholds of UP-algebras is introduced, some properties of them are discussed, and its generalizations are proved. Further, we discuss the relations between fuzzy UP-subalgebras (resp., fuzzy UP-filters, fuzzy UP-ideals, fuzzy strongly UP-ideals) with thresholds and its level subsets.

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1 Introduction

A fuzzy set in a nonempty set X is an arbitrary function from the set X into $[0,1]$ where $[0,1]$ is the unit segment of the real line. The concept of a fuzzy set in a nonempty set was first considered by Zadeh [14] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere. There are several kinds of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, bipolar-valued fuzzy sets etc.

After the introduction of the notion of fuzzy sets by Zadeh [14], several researches were conducted on the generalizations of the notion of fuzzy sets and application to many logical algebras such as: In 2005, Jun [4] introduced the notion of (α, β)-fuzzy subalgebras of BCK/BCI-algebras. In 2007, Jun [5] introduced the notion of fuzzy subalgebras with thresholds of BCK/BCI-algebras. In 2009, Saeid [11] introduced the notion of new fuzzy subalgebras with thresholds of BCK/BCI-algebras. Zhan et al. [15] introduced the notions of $(\in, \in \vee q)$-fuzzy p-ideals, $(\in, \in \vee q)$-fuzzy q-ideals and $(\in, \in \vee q)$-fuzzy a-ideals in BCI-algebras. In 2013, Li and Sun [7] introduced the notion of intuitionistic fuzzy implicative

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ideals with thresholds (λ, μ) of BCI-algebras. Zulfiqar [16] introduced the notion of fuzzy fantastic ideals in BCH-algebras. In 2014, Sun and Li [13] introduced the notion of intuitionistic fuzzy subalgebras with thresholds (λ, μ) and intuitionistic fuzzy ideals with thresholds (λ, μ) of BCI-algebras. Zulfiqar [17] introduced the notion of $(\overline{\alpha}, \overline{\beta})$-fuzzy fantastic ideals in BCH-algebras.

In this paper, we introduce the notion of fuzzy UP-subalgebras (resp., fuzzy UP-filters, fuzzy UP-ideals, fuzzy strongly UP-ideals) with thresholds of UP-algebras, investigate their properties, and generalize the notions. Also, the characterizations of fuzzy UP-subalgebras (resp., fuzzy UP-filters, fuzzy UP-ideals, fuzzy strongly UP-ideals) with thresholds are obtained. Further, we discuss the relations between fuzzy UP-subalgebras (resp., fuzzy UP-filters, fuzzy UP-ideals, fuzzy strongly UP-ideals) with thresholds and its level subsets.

2 Basic Results on UP-Algebras

Before we begin our study, we will introduce the definition of a UP-algebra.

An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* [2] where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms: for any $x, y, z \in A$,

(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,

(UP-2) $0 \cdot x = x$,

(UP-3) $x \cdot 0 = 0$, and

(UP-4) $x \cdot y = 0$ and $y \cdot x = 0$ imply $x = y$.

From [2], we know that the notion of UP-algebras is a generalization of KU-algebras.

Example 2.1. [2] Let X be a universal set. Define two binary operations $\cdot$ and $*$ on the power set of X by putting $A \cdot B = B \cap A'$ and $A * B = B \cup A'$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X), \cdot, \emptyset)$ and $(\mathcal{P}(X), *, X)$ are UP-algebras and we shall call it the *power UP-algebra of type 1* and the *power UP-algebra of type 2*, respectively.

In what follows, let A be a UP-algebra unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 2.2. *[2] In a UP-algebra A, the following properties hold: for any $x, y, z \in A$,*

(1) $x \cdot x = 0$,

(2) $x \cdot y = 0$ *and* $y \cdot z = 0$ *imply* $x \cdot z = 0$,

(3) $x \cdot y = 0$ *implies* $(z \cdot x) \cdot (z \cdot y) = 0$,

(4) $x \cdot y = 0$ *implies* $(y \cdot z) \cdot (x \cdot z) = 0$,

(5) $x \cdot (y \cdot x) = 0$,

(6) $(y \cdot x) \cdot x = 0$ *if and only if* $x = y \cdot x$, *and*

(7) $x \cdot (y \cdot y) = 0$.

Definition 2.3. [2] A subset S of A is called a *UP-subalgebra* of A if the constant 0 of A is in S, and $(S, \cdot, 0)$ itself forms a UP-algebra.

Iampan [2] proved the useful criteria that a nonempty subset S of a UP-algebra $A = (A, \cdot, 0)$ is a UP-subalgebra of A if and only if S is closed under the $\cdot$ multiplication on A.

Definition 2.4. [2, 12] A subset S of A is called a

(1) *UP-ideal* of A if it satisfies the following properties:

  (1) the constant 0 of A is in S, and
  (2) for any $x, y, z \in A, x \cdot (y \cdot z) \in S$ and $y \in S$ imply $x \cdot z \in S$.

(2) *UP-filter* of A if it satisfies the following properties:

  (1) the constant 0 of A is in S, and
  (2) for any $x, y \in A, x \cdot y \in S$ and $x \in S$ imply $y \in S$.

(3) *strongly UP-ideal* of A if it satisfies the following properties:

  (1) the constant 0 of A is in S, and
  (2) for any $x, y, z \in A, (z \cdot y) \cdot (z \cdot x) \in S$ and $y \in S$ imply $x \in S$.

Guntasow et al. [1] proved the generalization that the notion of UP-subalgebras is a generalization of UP-filters, the notion of UP-filters is a generalization of UP-ideals, and the notion of UP-ideals is a generalization of strongly UP-ideals. Moreover, they also proved that a UP-algebra A is the only one strongly UP-ideal of itself.

Definition 2.5. [14] A *fuzzy set* in a nonempty set X (or a fuzzy subset of X) is an arbitrary function from the set X into $[0, 1]$ where [0,1] is the unit segment of the real line.

Somjanta et al. [12] introduced the notion of fuzzy UP-subalgebras (resp., fuzzy UP-filters, fuzzy UP-ideals, fuzzy strongly UP-ideals) of UP-algebras as follows:

Definition 2.6. A fuzzy set f in A is called a

(1) *fuzzy UP-subalgebra* of A if for any $x, y \in A, f(x \cdot y) \geq \min\{f(x), f(y)\}$.

(2) *fuzzy UP-filter* of A if for any $x, y \in A$,

  (1) $f(0) \geq f(x)$, and
  (2) $f(y) \geq \min\{f(x \cdot y), f(x)\}$.

(3) *fuzzy UP-ideal* of A if for any $x, y, z \in A$,

  (1) $f(0) \geq f(x)$, and
  (2) $f(x \cdot z) \geq \min\{f(x \cdot (y \cdot z)), f(y)\}$.

(4) *fuzzy strongly UP-ideal* of A if for any $x, y, z \in A$,

  (1) $f(0) \geq f(x)$, and
  (2) $f(x) \geq \min\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$.

They also proved that the notion of fuzzy UP-subalgebras is a generalization of fuzzy UP-filters, the notion of fuzzy UP-filters is a generalization of fuzzy UP-ideals, and the notion of fuzzy UP-ideals is a generalization of fuzzy strongly UP-ideals.

Definition 2.7. [8] Let X and Y be any two nonempty sets and let $f\colon X \to Y$ be any function. A fuzzy set μ in X is called *f-invariant* if $f(x) = f(y)$ implies $\mu(x) = \mu(y)$ for all $x, y \in X$.

Definition 2.8. [2] Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras. A mapping f from A to B is called a *UP-homomorphism* if

$$f(x \cdot y) = f(x) * f(y) \text{ for all } x, y \in A.$$

A UP-homomorphism $f \colon A \to B$ is called a

(1) *UP-endomorphism* of A if $B = A$,

(2) *UP-epimorphism* if it is surjective,

(3) *UP-monomorphism* if it is injective, and

(4) *UP-isomorphism* if it is bijective.

Iampan [2] proved that $f(0_A) = 0_B$.

3 Fuzzy UP-Subalgebras (resp., Fuzzy UP-Filters, Fuzzy UP-Ideals, Fuzzy Strongly UP-Ideals) with Thresholds

In this section, we introduce the notion of fuzzy UP-subalgebras (resp., fuzzy UP-filters, fuzzy UP-ideals, fuzzy strongly UP-ideals) with thresholds of UP-algebras, investigate their properties, and generalize the notions.

Definition 3.1. A fuzzy set f in A is called a *fuzzy UP-subalgebra with thresholds* ε *and* δ of A where $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$ if for any $x, y \in A$,

$$\max\{f(x \cdot y), \varepsilon\} \geq \min\{f(x), f(y), \delta\}.$$

Example 3.2. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	3	0
2	0	1	0	0
3	0	1	3	0

Then $(A, \cdot, 0)$ is a UP-algebra. We defined a fuzzy set $f \colon A \to [0, 1]$ as follows:

$$f(0) = 0.6, f(1) = 0.7, f(2) = 0.3, \text{ and } f(3) = 0.2.$$

Then f is a fuzzy UP-subalgebra with thresholds $\varepsilon = 0.8$ and $\delta = 0.9$ of A.

Lemma 3.3. *If f is a fuzzy UP-subalgebra with thresholds ε and δ of A, then* $\max\{f(0), \varepsilon\} \geq \min\{f(x), \delta\}$ *for all $x \in A$. In particular, if there exists $y \in A$ such that $f(y) > \delta$, then $f(0) > \varepsilon$.*

Proof. For all $x \in A$,

$$\begin{aligned} \max\{f(0), \varepsilon\} &= \max\{f(x \cdot x), \varepsilon\} && \text{(Proposition 2.2 (1))} \\ &\geq \min\{f(x), f(x), \delta\} \\ &= \min\{f(x), \delta\}. \end{aligned}$$

If there exists $y \in A$ such that $f(y) > \delta$, then

$$\max\{f(0), \varepsilon\} \geq \min\{f(y), \delta\} = \delta > \varepsilon.$$

Thus $f(0) = \max\{f(0), \varepsilon\} > \varepsilon$. □

Definition 3.4. A fuzzy set f in A is called a *fuzzy UP-filter with thresholds* ε *and* δ of A where $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$ if it satisfies the following properties: for any $x, y \in A$,

(1) $\max\{f(0), \varepsilon\} \geq \min\{f(x), \delta\}$, and

(2) $\max\{f(y), \varepsilon\} \geq \min\{f(x \cdot y), f(x), \delta\}$.

Example 3.5. Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	0	0
2	0	1	0	0	0
3	0	1	2	0	4
4	0	1	2	3	0

Then $(A, \cdot, 0)$ is a UP-algebra. We defined a fuzzy set $f : A \to [0, 1]$ as follows:

$$f(0) = 0.9, f(1) = 0.1, f(2) = 0.5, f(3) = 0.4, \text{ and } f(4) = 0.4.$$

Then f is a fuzzy UP-filter with thresholds $\varepsilon = 0.8$ and $\delta = 0.9$ of A.

Definition 3.6. A fuzzy set f in A is called a *fuzzy UP-ideal with thresholds* ε *and* δ of A where $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$, if it satisfies the following properties: for any $x, y, z \in A$,

(1) $\max\{f(0), \varepsilon\} \geq \min\{f(x), \delta\}$, and

(2) $\max\{f(x \cdot z), \varepsilon\} \geq \min\{f(x \cdot (y \cdot z)), f(y), \delta\}$.

Example 3.7. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	0	1	0	0
3	0	1	3	0

Then $(A, \cdot, 0)$ is a UP-algebra. We defined a fuzzy set $f \colon A \to [0, 1]$ as follows:

$$f(0) = 0, f(1) = 0.2, f(2) = 0.1, \text{ and } f(3) = 0.3.$$

Then f is a fuzzy UP-ideal with thresholds $\varepsilon = 0.4$ and $\delta = 0.9$ of A.

Definition 3.8. A fuzzy set f in A is called a *fuzzy strongly UP-ideal with thresholds* ε *and* δ of A where $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$ if it satisfies the following properties: for any $x, y, z \in A$,

(1) $\max\{f(0), \varepsilon\} \geq \min\{f(x), \delta\}$, and

(2) $\max\{f(x), \varepsilon\} \geq \min\{f((z \cdot y) \cdot (z \cdot x)), f(y), \delta\}$.

Example 3.9. Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	0	4
2	0	1	0	0	0
3	0	1	2	0	4
4	0	1	2	3	0

Then $(A, \cdot, 0)$ is a UP-algebra. We defined a fuzzy set $f : A \to [0, 1]$ as follow:

$$f(0) = 0.3, f(1) = 0.4, f(2) = 0, f(3) = 0.1, \text{ and } f(4) = 0.$$

Then f is a fuzzy strongly UP-ideal with thresholds $\varepsilon = 0.5$ and $\delta = 0.8$ of A.

Theorem 3.10. *Every fuzzy strongly UP-ideal with thresholds ε and δ of A is a fuzzy UP-ideal with thresholds ε and δ.*

Proof. Assume that f is a fuzzy strongly UP-ideal with thresholds ε and δ of A. Then $\max\{f(0), \varepsilon\} \geq \min\{f(x), \delta\}$ for all $x \in A$. Let $x, y, z \in A$.

Case 1: $f(0) \leq \varepsilon$. Then $f(x) \leq \varepsilon$ for all $x \in A$. Thus

$$\begin{aligned} \max\{f(x \cdot z), \varepsilon\} &= \varepsilon \\ &\geq f(y) \\ &\geq \min\{f(x \cdot (y \cdot z)), f(y), \delta\}. \end{aligned}$$

Case 2: $f(0) > \varepsilon$. Then

$$\begin{aligned} \max\{f(x \cdot z), \varepsilon\} &\geq \min\{f((z \cdot y) \cdot (z \cdot (x \cdot z))), f(y), \delta\} \\ &= \min\{f((z \cdot y) \cdot 0), f(y), \delta\} && \text{(Proposition 2.2 (5))} \\ &= \min\{f(0), f(y), \delta\}. && \text{((UP-3))} \end{aligned}$$

If $\min\{f(0), f(y), \delta\} = f(y)$ or δ, we obtain immediately that $\max\{f(x \cdot z), \varepsilon\} \geq \min\{f(x \cdot (y \cdot z)), f(y), \delta\}$. Assume that $\min\{f(0), f(y), \delta\} = f(0)$. Then

$$\begin{aligned} \min\{f(0), f(y), \delta\} &= f(0) \\ &= \max\{f(0), \varepsilon\} \\ &\geq \min\{f(y), \delta\} \\ &\geq \min\{f(x \cdot (y \cdot z)), f(y), \delta\}. \end{aligned}$$

Hence, f is a fuzzy UP-ideal with thresholds ε and δ of A. □

Example 3.11. Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	0	4
2	0	1	0	0	0
3	0	1	2	0	4
4	0	1	2	3	0

Then $(A, \cdot, 0)$ is a UP-algebra. We defined a fuzzy set $f : A \to [0, 1]$ as follow:

$$f(0) = 1, f(1) = 0.2, f(2) = 0.1, f(3) = 0.2, \text{ and } f(4) = 0.9.$$

Then f is a fuzzy UP-ideal with thresholds $\varepsilon = 0.5$ and $\delta = 0.9$ of A. Since $\max\{f(2), \varepsilon\} = \max\{0.1, 0.5\} = 0.5 \ngeq 0.9 = \min\{1, 1, 0.9\} = \min\{f((2 \cdot 0) \cdot (2 \cdot 2)), f(0), \delta\}$, we have f is not a fuzzy strongly UP-ideal with thresholds $\varepsilon = 0.5$ and $\delta = 0.9$ of A.

Theorem 3.12. *Every fuzzy UP-ideal with thresholds ε and δ of A is a fuzzy UP-filter with thresholds ε and δ.*

Proof. Assume that f is a fuzzy UP-ideal with thresholds ε and δ of A. Then $\max\{f(0), \varepsilon\} \geq \min\{f(x), \delta\}$ for all $x \in A$. Let $x, y \in A$. Then

$$\begin{aligned} \max\{f(y), \varepsilon\} &= \max\{f(0 \cdot y), \varepsilon\} && \text{((UP-2))} \\ &\geq \min\{f(0 \cdot (x \cdot y)), f(x), \delta\} \\ &= \min\{f(x \cdot y), f(x), \delta\}. && \text{((UP-2))} \end{aligned}$$

Hence, f is a fuzzy UP-filter with thresholds ε and δ of A. □

Example 3.13. Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	3
3	0	1	2	0	3
4	0	1	2	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We defined a fuzzy set $f \colon A \to [0, 1]$ as follow:

$$f(0) = 0.9, f(1) = 0.8, f(2) = 0.7, f(3) = 0.5, \text{ and } f(4) = 0.5.$$

Then f is a fuzzy UP-filter with thresholds $\varepsilon = 0.2$ and $\delta = 0.9$ of A. Since $\max\{f(3 \cdot 4), \varepsilon\} = \max\{0.5, 0.2\} = 0.5 \ngeq 0.7 = \min\{0.9, 0.7, 0.9\} = \min\{f(3 \cdot (2 \cdot 4)), f(2), \delta\}$, we have f is not a fuzzy UP-ideal with thresholds $\varepsilon = 0.2$ and $\delta = 0.9$ of A.

Theorem 3.14. *Every fuzzy UP-filter with thresholds ε and δ of A is a fuzzy UP-subalgebra with thresholds ε and δ.*

Proof. Assume that f is a fuzzy UP-filter with thresholds ε and δ of A. Then $\max\{f(0), \varepsilon\} \geq \min\{f(x), \delta\}$ for all $x \in A$. Let $x, y \in A$. Then

Case 1: $f(0) \leq \varepsilon$. Then $f(x) \leq \varepsilon$ for all $x \in A$. Thus

$$\begin{aligned} \max\{f(x \cdot y), \varepsilon\} &= \varepsilon \\ &\geq f(x) \\ &\geq \min\{f(x), f(y), \delta\}. \end{aligned}$$

Case 2: $f(0) > \varepsilon$. Then

$$\begin{aligned} \max\{f(x \cdot y), \varepsilon\} &\geq \min\{f(y \cdot (x \cdot y)), f(y), \delta\} \\ &= \min\{f(0), f(y), \delta\}. && \text{(Proposition 2.2 (5))} \end{aligned}$$

If $\min\{f(0), f(y), \delta\} = f(y)$ or δ, we obtain immediately that $\max\{f(x \cdot y), \varepsilon\} \geq \min\{f(x), f(y), \delta\}$. Assume that $\min\{f(0), f(y), \delta\} = f(0)$. Then

$$\begin{aligned} \min\{f(0), f(y), \delta\} &= f(0) \\ &= \max\{f(0), \varepsilon\} \\ &\geq \min\{f(y), \delta\} \\ &\geq \min\{f(x), f(y), \delta\}. \end{aligned}$$

Hence, f is a fuzzy UP-subalgebra with thresholds ε and δ of A. □

Example 3.15. Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	3
3	0	1	2	0	3
4	0	1	2	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We defined a fuzzy set $f: A \to [0, 1]$ as follow:

$$f(0) = 0.8, f(1) = 0.7, f(2) = 0.4, f(3) = 0.3, \text{ and } f(4) = 0.2.$$

Then f is a fuzzy UP-subalgebra with thresholds $\varepsilon = 0.2$ and $\delta = 0.9$ of A. Since $\max\{f(4), \varepsilon\} = \max\{0.2, 0.2\} = 0.2 \ngeq 0.3 = \min\{0.3, 0.3, 0.9\} = \min\{f(3 \cdot 4), f(3), \delta\}$, we have f is not a fuzzy UP-filter with thresholds $\varepsilon = 0.2$ and $\delta = 0.9$ of A.

By Definition 2.6, we see that a fuzzy UP-subalgebra (resp., fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) is a fuzzy UP-subalgebra (resp., fuzzy UP-filter, fuzzy UP-ideal, fuzzy strongly UP-ideal) with thresholds 0 and 1. By Theorem 3.10, 3.12, and 3.14 and Example 3.11, 3.13, and 3.15, we have that the notion of fuzzy UP-subalgebras with thresholds ε and δ is a generalization of fuzzy UP-filters with thresholds ε and δ, the notion of fuzzy UP-filters with thresholds ε and δ is a generalization of fuzzy UP-ideals with thresholds ε and δ, and the notion of fuzzy UP-ideals with thresholds ε and δ is a generalization of fuzzy strongly UP-ideals with thresholds ε and δ.

Theorem 3.16. *Let $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. If a fuzzy set f in A is constant, then it is a fuzzy strongly UP-ideal with thresholds ε and δ of A.*

Proof. Assume that f is a constant fuzzy set in A and let $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. Then $f(x) = f(0)$ for all $x \in A$. Let $x \in A$.

$$\max\{f(0), \varepsilon\} \geq f(0) = f(x) \geq \min\{f(x), \delta\}.$$

Let $x, y, z \in A$. Then

$$\begin{aligned} \max\{f(x), \varepsilon\} &\geq f(x) \\ &= \min\{f((z \cdot y) \cdot (z \cdot x)), f(y)\} \\ &\geq \min\{f((z \cdot y) \cdot (z \cdot x)), f(y), \delta\}. \end{aligned}$$

Hence, f is a fuzzy strongly UP-ideal with thresholds ε and δ of A. □

Theorem 3.17. *Let $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. If a fuzzy set f in A is such that $f(x) \leq \varepsilon$ for all $x \in A$, then it is a fuzzy strongly UP-ideal (resp. fuzzy UP-ideal, fuzzy UP-filter and fuzzy UP-subalgebra) with thresholds ε and δ of A.*

Proof. (1) Let $x \in A$. Then

$$\max\{f(0), \varepsilon\} = \varepsilon \geq f(x) = \min\{f(x), \delta\}.$$

Let $x, y, z \in A$. Then

$$\begin{aligned} \max\{f(x), \varepsilon\} &= \varepsilon \\ &\geq \min\{f((z \cdot y) \cdot (z \cdot x)), f(y)\} \\ &= \min\{f((z \cdot y) \cdot (z \cdot x)), f(y), \delta\}. \end{aligned}$$

Hence, f is a fuzzy strongly UP-ideal with thresholds ε and δ of A.

(2) Let $x, y, z \in A$. Then

$$\begin{aligned}\max\{f(x \cdot z), \varepsilon\} &= \varepsilon \\ &\geq \min\{f(x \cdot (y \cdot z)), f(y)\} \\ &= \min\{f(x \cdot (y \cdot z)), f(y), \delta\}.\end{aligned}$$

Hence, f is a fuzzy UP-ideal with thresholds ε and δ of A.

(3) Let $x, y \in A$. Then

$$\begin{aligned}\max\{f(y), \varepsilon\} &= \varepsilon \\ &\geq \min\{f(x \cdot y), f(x)\} \\ &= \min\{f(x \cdot y), f(x), \delta\}.\end{aligned}$$

Hence, f is a fuzzy UP-filter with thresholds ε and δ of A.

(4) Let $x, y \in A$. Then

$$\begin{aligned}\max\{f(x \cdot y), \varepsilon\} &= \varepsilon \\ &\geq \min\{f(x), f(y)\} \\ &= \min\{f(x), f(y), \delta\}.\end{aligned}$$

Hence, f is a fuzzy UP-subalgebra with thresholds ε and δ of A. □

Theorem 3.18. *Let $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. If a fuzzy set f in A with $\varepsilon \leq f(x) \leq \delta$ is a fuzzy strongly UP-ideal with thresholds ε and δ of A, then it is constant.*

Proof. For all $x \in A$,

$$f(0) = \max\{f(0), \varepsilon\} \geq \min\{f(x), \delta\} = f(x)$$

and

$$\begin{aligned}f(x) &= \max\{f(x), \varepsilon\} \\ &\geq \min\{f((x \cdot 0) \cdot (x \cdot x)), f(0), \delta\} \\ &= \min\{f((x \cdot 0) \cdot 0), f(0), \delta\} && \text{(Proposition 2.2 (1))} \\ &= \min\{f(0), f(0), \delta\} && \text{((UP-3))} \\ &= \min\{f(0), \delta\} \\ &= f(0).\end{aligned}$$

Hence, $f(x) = f(0)$ for all $x \in A$, so f is constant. □

Theorem 3.19. *Let $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. If a fuzzy set f in A is such that $f(x) \geq \delta$ for all $x \in A$, then it is a fuzzy strongly UP-ideal (resp. fuzzy UP-ideal, fuzzy UP-filter and fuzzy UP-subalgebra) with thresholds ε and δ of A.*

Proof. (1) Let $x \in A$. Then

$$\max\{f(0), \varepsilon\} = f(0) \geq \delta = \min\{f(x), \delta\}.$$

Let $x, y, z \in A$. Then

$$\begin{aligned}\max\{f(x), \varepsilon\} &= f(x) \\ &\geq \delta \\ &= \min\{f((z \cdot y) \cdot (z \cdot x)), f(y), \delta\}.\end{aligned}$$

Hence, f is a fuzzy strongly UP-ideal with thresholds ε and δ of A.

(2) Let $x, y, z \in A$. Then

$$\begin{aligned}\max\{f(x \cdot z), \varepsilon\} &= f(x \cdot z)\\ &\geq \delta\\ &= \min\{f(x \cdot (y \cdot z)), f(y), \delta\}.\end{aligned}$$

Hence, f is a fuzzy UP-ideal with thresholds ε and δ of A.

(3) Let $x, y \in A$. Then

$$\begin{aligned}\max\{f(y), \varepsilon\} &= f(y)\\ &\geq \delta\\ &= \min\{f(x \cdot y), f(x), \delta\}.\end{aligned}$$

Hence, f is a fuzzy UP-filter with thresholds ε and δ of A.

(4) Let $x, y \in A$. Then

$$\begin{aligned}\max\{f(x \cdot y), \varepsilon\} &= f(x \cdot y)\\ &\geq \delta\\ &= \min\{f(x), f(y), \delta\}.\end{aligned}$$

Hence, f is a fuzzy UP-subalgebra with thresholds ε and δ of A. □

Corollary 3.20. *Let $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. Then a fuzzy set f in A with $\varepsilon \leq f(x) \leq \delta$ is a fuzzy strongly UP-ideal with thresholds ε and δ of A if and only if it is constant.*

Proof. It is straightforward by Theorem 3.16 and 3.18. □

4 Upper t-Level Subsets of a Fuzzy Set

In this section, we discuss the relations between fuzzy UP-subalgebras (resp., fuzzy UP-filters, fuzzy UP-ideals, fuzzy strongly UP-ideals) with thresholds and its level subsets.

Let f be a fuzzy set in A. For any $t \in [0, 1]$, the set

$$U(f; t) = \{x \in A \mid f(x) \geq t\}$$

is called an *upper t-level subset* [12] of f.

Theorem 4.1. *Let f be a fuzzy set in A and $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. Then f is a fuzzy UP-subalgebra with thresholds ε and δ of A if and only if for all $t \in (\varepsilon, \delta], U(f; t)$ is a UP-subalgebra of A if $U(f; t)$ is nonempty.*

Proof. Assume that f is a fuzzy UP-subalgebra with thresholds ε and δ of A. Let $t \in (\varepsilon, \delta]$ be such that $U(f; t) \neq \emptyset$ and let $x, y \in U(f; t)$. Then $f(x) \geq t, f(y) \geq t$, and $\delta \geq t$. Thus t is a lower bound of $\{f(x), f(y), \delta\}$. Since f is a fuzzy UP-subalgebra with thresholds ε and δ of A, we have

$$\begin{aligned}\max\{f(x \cdot y), \varepsilon\} &\geq \min\{f(x), f(y), \delta\}\\ &\geq t\\ &> \varepsilon.\end{aligned}$$

Thus $\max\{f(x \cdot y), \varepsilon\} = f(x \cdot y)$. Since $\max\{f(x \cdot y), \varepsilon\} \geq t$, we get $f(x \cdot y) \geq t$. Thus $x \cdot y \in U(f; t)$. Hence, $U(f; t)$ is a UP-subalgebra of A.

Conversely, assume that for all $t \in (\varepsilon, \delta]$, $U(f;t)$ is a UP-subalgebra of A if $U(f;t)$ is nonempty. Let $x, y \in A$. Then $f(x), f(y) \in [0,1]$. Choose $t = \min\{f(x), f(y)\}$. Then $f(x) \geq t$ and $f(y) \geq t$. Thus $x, y \in U(f;t) \neq \emptyset$. By assumption, we have $U(f;t)$ is a UP-subalgebra of A. So $x \cdot y \in U(f;t)$, that is, $f(x \cdot y) \geq t = \min\{f(x), f(y)\}$. Thus

$$\begin{aligned}\max\{f(x \cdot y), \varepsilon\} &\geq f(x \cdot y)\\ &\geq \min\{f(x), f(y)\}\\ &\geq \min\{f(x), f(y), \delta\}.\end{aligned}$$

Hence, f is a UP-subalgebra with thresholds ε and δ of A. □

Theorem 4.2. *Let f be a fuzzy set in A and $\varepsilon, \delta \in [0,1]$ with $\varepsilon < \delta$. Then f is a fuzzy UP-filter with thresholds ε and δ of A if and only if for all $t \in (\varepsilon, \delta], U(f;t)$ is a UP-filter of A if $U(f;t)$ is nonempty.*

Proof. Assume that f is a fuzzy UP-filter with thresholds ε and δ of A. Let $t \in (\varepsilon, \delta]$ be such that $U(f;t) \neq \emptyset$ and let $a \in U(f;t)$. Then $f(a) \geq t$ and $\delta \geq t$. Thus t is a lower bound of $\{f(a), \delta\}$. Since f is a fuzzy UP-filter with thresholds ε and δ of A, we have

$$\begin{aligned}\max\{f(0), \varepsilon\} &\geq \min\{f(a), \delta\}\\ &\geq t\\ &> \varepsilon.\end{aligned}$$

Thus $\max\{f(0), \varepsilon\} = f(0)$. Since $\max\{f(0), \varepsilon\} \geq t$, we get $f(0) \geq t$. Thus $0 \in U(f;t)$. Next, let $x, y \in A$ be such that $x \cdot y \in U(f;t)$ and $x \in U(f;t)$. Then $f(x \cdot y) \geq t, f(x) \geq t$, and $\delta \geq t$. Thus t is a lower bound of $\{f(x \cdot y), f(x), \delta\}$. Since f is a fuzzy UP-filter with thresholds ε and δ of A, we have

$$\begin{aligned}\max\{f(y), \varepsilon\} &\geq \min\{f(x \cdot y), f(x), \delta\}\\ &\geq t\\ &> \varepsilon.\end{aligned}$$

Thus $\max\{f(y), \varepsilon\} = f(y)$. Since $\max\{f(y), \varepsilon\} \geq t$, we get $f(y) \geq t$. Thus $y \in U(f;t)$. Hence, $U(f;t)$ is a UP-filter of A.

Conversely, assume that for all $t \in (\varepsilon, \delta]$, $U(f;t)$ is a UP-filter of A if $U(f;t)$ is nonempty. Let $x \in A$. Then $f(x) \in [0,1]$. Choose $t = f(x)$. Then $f(x) \geq t$. Thus $x \in U(f;t) \neq \emptyset$. By assumption, we have $U(f;t)$ is a UP-filter of A. So $0 \in U(f;t)$, that is, $f(0) \geq t = f(x)$, so

$$\begin{aligned}\max\{f(0), \varepsilon\} &\geq f(0)\\ &\geq f(x)\\ &\geq \min\{f(x), \delta\}.\end{aligned}$$

Next, let $x, y \in A$. Then $f(x \cdot y), f(x) \in [0,1]$. Choose $t = \min\{f(x \cdot y), f(x)\}$. Then $f(x \cdot y) \geq t$ and $f(x) \geq t$. Thus $x \cdot y, x \in U(f;t) \neq \emptyset$. By assumption, we have $U(f;t)$ is a UP-filter of A. So $y \in U(f;t)$, that is, $f(y) \geq t = \min\{f(x \cdot y), f(x)\}$. Thus

$$\begin{aligned}\max\{f(y), \varepsilon\} &\geq f(y)\\ &\geq \min\{f(x \cdot y), f(x)\}\\ &\geq \min\{f(x \cdot y), f(x), \delta\}.\end{aligned}$$

f is a fuzzy UP-filter with thresholds ε and δ of A. □

Theorem 4.3. *Let f be a fuzzy set in A and $\varepsilon, \delta \in [0,1]$ with $\varepsilon < \delta$. Then f is a fuzzy UP-ideal with thresholds ε and δ of A if and only if for all $t \in (\varepsilon, \delta], U(f;t)$ is a UP-ideal of A if $U(f;t)$ is nonempty.*

Proof. Assume that f is a fuzzy UP-ideal with thresholds ε and δ of A. Let $t \in (\varepsilon, \delta]$ be such that $U(f;t) \neq \emptyset$ and let $a \in U(f;t)$. Then $f(a) \geq t$ and $\delta \geq t$. Thus t is a lower bound of $\{f(a), \delta\}$. Since f is a fuzzy UP-ideal with thresholds ε and δ of A, we have

$$\begin{aligned} \max\{f(0), \varepsilon\} &\geq \min\{f(a), \delta\} \\ &\geq t \\ &> \varepsilon. \end{aligned}$$

Thus $\max\{f(0), \varepsilon\} = f(0)$. Since $\max\{f(0), \varepsilon\} \geq t$, we get $f(0) \geq t$. Thus $0 \in U(f;t)$. Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in U(f;t)$ and $y \in U(f;t)$. Then $f(x \cdot (y \cdot z)) \geq t, f(y) \geq t$, and $\delta \geq t$. Thus t is a lower bound of $\{f(x \cdot (y \cdot z)), f(y), \delta\}$. Since f is a fuzzy UP-ideal with thresholds ε and δ of A, we have

$$\begin{aligned} \max\{f(x \cdot z), \varepsilon\} &\geq \min\{f(x \cdot (y \cdot z)), f(y), \delta\} \\ &\geq t \\ &> \varepsilon. \end{aligned}$$

Thus $\max\{f(x \cdot z), \varepsilon\} = f(x \cdot z)$. Since $\max\{f(x \cdot z), \varepsilon\} \geq t$, we get $f(x \cdot z) \geq t$. Thus $x \cdot z \in U(f;t)$. Hence, $U(f;t)$ is a UP-ideal of A.

Conversely, assume that for all $t \in (\varepsilon, \delta]$, $U(f;t)$ is a UP-ideal of A if $U(f;t)$ is nonempty. Let $x \in A$. Then $f(x) \in [0,1]$. Choose $t = f(x)$. Then $f(x) \geq t$. Thus $x \in U(f;t) \neq \emptyset$. By assumption, we have $U(f;t)$ is a UP-ideal of A. So $0 \in U(f;t)$, that is, $f(0) \geq t = f(x)$, so

$$\begin{aligned} \max\{f(0), \varepsilon\} &\geq f(0) \\ &\geq f(x) \\ &\geq \min\{f(x), \delta\}. \end{aligned}$$

Next, let $x, y, z \in A$. Then $f(x \cdot (y \cdot z)), f(y) \in [0,1]$. Choose $t = \min\{f(x \cdot (y \cdot z)), f(y)\}$. Then $f(x \cdot (y \cdot z)) \geq t$ and $f(y) \geq t$. Thus $x \cdot (y \cdot z), y \in U(f;t) \neq \emptyset$. By assumption, we have $U(f;t)$ is a UP-ideal of A. So $x \cdot z \in U(f;t)$, that is, $f(x \cdot z) \geq t = \min\{f(x \cdot (y \cdot z)), f(y)\}$. Thus

$$\begin{aligned} \max\{f(x \cdot z), \varepsilon\} &\geq f(x \cdot z) \\ &\geq \min\{f(x \cdot (y \cdot z)), f(y)\} \\ &\geq \min\{f(x \cdot (y \cdot z)), f(y), \delta\}. \end{aligned}$$

Hence, f is a fuzzy UP-ideal with thresholds ε and δ of A. □

Theorem 4.4. *Let f be a fuzzy set in A and $\varepsilon, \delta \in [0,1]$ with $\varepsilon < \delta$. Then f is a fuzzy strongly UP-ideal with thresholds ε and δ of A if and only if for all $t \in (\varepsilon, \delta], U(f;t)$ is a strongly UP-ideal of A if $U(f;t)$ is nonempty.*

Proof. Assume that f is a fuzzy strongly UP-ideal with thresholds ε and δ of A. Let $t \in (\varepsilon, \delta]$ be such that $U(f;t) \neq \emptyset$ and let $a \in U(f;t)$. Then $f(a) \geq t$ and $\delta \geq t$. Thus t is a lower bound of $\{f(a), \delta\}$. Since f is a fuzzy strongly UP-ideal with thresholds ε and δ of A, we have

$$\begin{aligned} \max\{f(0), \varepsilon\} &\geq \min\{f(a), \delta\} \\ &\geq t \\ &> \varepsilon. \end{aligned}$$

Thus $\max\{f(0), \varepsilon\} = f(0)$. Since $\max\{f(0), \varepsilon\} \geq t$, we get $f(0) \geq t$. Thus $0 \in U(f;t)$. Next, let $x, y, z \in A$ be such that $(z \cdot y) \cdot (z \cdot x) \in U(f;t)$ and $y \in U(f;t)$. Then $f((z \cdot y) \cdot (z \cdot x)) \geq t, f(y) \geq t$, and $\delta \geq t$. Thus t is a lower bound of $\{f((z \cdot y) \cdot (z \cdot x)), f(y), \delta\}$. Since f is a fuzzy strongly UP-ideal with thresholds ε and δ of A, we have

$$\begin{aligned}\max\{f(x), \varepsilon\} &\geq \min\{f((z \cdot y) \cdot (z \cdot x)), f(y), \delta\} \\ &\geq t \\ &> \varepsilon.\end{aligned}$$

Thus $\max\{f(x), \varepsilon\} = f(x)$. Since $\max\{f(x), \varepsilon\} \geq t$, we get $f(x) \geq t$. Thus $x \in U(f;t)$. Hence, $U(f;t)$ is a strongly UP-ideal of A.

Conversely, assume that for all $t \in (\varepsilon, \delta]$, $U(f;t)$ is a strongly UP-ideal of A if $U(f;t)$ is nonempty. Let $x \in A$. Then $f(x) \in [0,1]$. Choose $t = f(x)$. Then $f(x) \geq t$. Thus $x \in U(f;t) \neq \emptyset$. By assumption, we have $U(f;t)$ is a strongly UP-ideal of A. So $0 \in U(f;t)$, that is, $f(0) \geq t = f(x)$, so

$$\begin{aligned}\max\{f(0), \varepsilon\} &\geq f(0) \\ &\geq f(x) \\ &\geq \min\{f(x), \delta\}.\end{aligned}$$

Next, let $x, y, z \in A$. Then $f((z \cdot y) \cdot (z \cdot x)), f(y) \in [0,1]$. Choose $t = \min\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$. Then $f((z \cdot y) \cdot (z \cdot x)) \geq t$ and $f(y) \geq t$. Thus $(z \cdot y) \cdot (z \cdot x), y \in U(f;t) \neq \emptyset$. By assumption, we have $U(f;t)$ is a strongly UP-ideal of A. So $x \in U(f;t)$, that is, $f(x) \geq t = \min\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$. Thus

$$\begin{aligned}\max\{f(x), \varepsilon\} &\geq f(x) \\ &\geq \min\{f((z \cdot y) \cdot (z \cdot x)), f(y)\} \\ &\geq \min\{f((z \cdot y) \cdot (z \cdot x)), f(y), \delta\}.\end{aligned}$$

Hence, f is a fuzzy strongly UP-ideal with thresholds ε and δ of A. □

5 Image and Preimage of a Fuzzy Set

Definition 5.1. [3] Let f be a function from a nonempty set X to a nonempty set Y. If μ is a fuzzy set in X, then the fuzzy set β in Y defined by

$$\beta(y) = \begin{cases} \sup\{\mu(t)\}_{t \in f^{-1}(y)} & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{if otherwise} \end{cases}$$

is said to be the *image of* μ *under* f. Similarly, if β is a fuzzy set in Y, then the fuzzy set $\mu = \beta \circ f$ in X (i.e., the fuzzy set defined by $\mu(x) = \beta(f(x))$ for all $x \in X$) is called the *preimage of* β *under* f.

Definition 5.2. [10] A fuzzy set f in A has sup *property* if for any nonempty subset T of A, there exists $t_0 \in T$ such that $f(t_0) = \sup\{f(t)\}_{t \in T}$.

Lemma 5.3. *[9] Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras and let $f\colon A \to B$ be a UP-epimorphism. Let μ be an f-invariant fuzzy set in A with* sup *property. For any $a, b \in B$, there exist $a_0 \in f^{-1}(a)$ and $b_0 \in f^{-1}(b)$ such that $\beta(a) = \mu(a_0), \beta(b) = \mu(b_0)$, and $\beta(a * b) = \mu(a_0 \cdot b_0)$.*

Theorem 5.4. *Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras and let $f\colon A \to B$ be a UP-epimorphism. Then the following statements hold:*

(1) if μ is an f-invariant fuzzy UP-subalgebra with thresholds ε and δ of A with sup *property, then β is a fuzzy UP-subalgebra with thresholds ε and δ of B,*

(2) if μ is an f-invariant fuzzy UP-filter with thresholds ε and δ of A with sup *property, then β is a fuzzy UP-filter with thresholds ε and δ of B,*

(3) if μ is an f-invariant fuzzy UP-ideal with thresholds ε and δ of A with sup *property, then β is a fuzzy UP-ideal with thresholds ε and δ of B, and*

(4) if μ is an f-invariant fuzzy strongly UP-ideal with thresholds ε and δ of A with sup *property, then β is a fuzzy strongly UP-ideal with thresholds ε and δ of B.*

Proof. (1) Assume that μ is an f-invariant fuzzy UP-subalgebra with thresholds ε and δ of A with sup property. Let $a, b \in B$. By Lemma 5.3, there exist $a_0 \in f^{-1}(a)$ and $b_0 \in f^{-1}(b)$ such that $\beta(a) = \mu(a_0), \beta(b) = \mu(b_0)$, and $\beta(a * b) = \mu(a_0 \cdot b_0)$. Thus

$$\begin{aligned}\max\{\beta(a * b), \varepsilon\} &= \max\{\mu(a_0 \cdot b_0), \varepsilon\}\\ &\geq \min\{\mu(a_0), \mu(b_0), \delta\}\\ &= \min\{\beta(a), \beta(b), \delta\}.\end{aligned}$$

Hence, β is a fuzzy UP-subalgebra with thresholds ε and δ of B.

(2) Assume that μ is an f-invariant fuzzy UP-filter with thresholds ε and δ of A with sup property. Since $f(0_A) = 0_B$, we have $f^{-1}(0_B) \neq \emptyset$. Then there exists $x_1 \in f^{-1}(0_B)$ such that $\mu(x_1) = \beta(0_B)$. Thus $f(x_1) = 0_B = f(0_A)$, so $\mu(x_1) = \mu(0_A)$ because μ is f-invariant. Hence, $\mu(0_A) = \beta(0_B)$. Let $y \in B$. Since f is surjective, we have $f^{-1}(y) \neq \emptyset$. Then there exists $x \in f^{-1}(y)$ such that $\mu(x) = \beta(y)$, so

$$\begin{aligned}\max\{\beta(0_B), \varepsilon\} &= \max\{\mu(0_A), \varepsilon\}\\ &\geq \min\{\mu(x), \delta\}\\ &= \min\{\beta(y), \delta\}.\end{aligned}$$

Next, let $a, b \in B$. By Lemma 5.3, there exist $a_0 \in f^{-1}(a)$ and $b_0 \in f^{-1}(b)$ such that $\mu(a_0) = \beta(a), \mu(b_0) = \beta(b)$, and $\mu(a_0 \cdot b_0) = \beta(a * b)$. Thus

$$\begin{aligned}\max\{\beta(b), \varepsilon\} &= \max\{\mu(b_0), \varepsilon\}\\ &\geq \min\{\mu(a_0 \cdot b_0), \mu(a_0), \delta\}\\ &= \min\{\beta(a * b), \beta(a), \delta\}.\end{aligned}$$

Hence, β is a fuzzy UP-filter with thresholds ε and δ of B.

(3) Assume that μ is an f-invariant fuzzy UP-ideal with thresholds ε and δ of A with sup property. Since $f(0_A) = 0_B$, we have $f^{-1}(0_B) \neq \emptyset$. Then there exists $x_1 \in f^{-1}(0_B)$ such that $\mu(x_1) = \beta(0_B)$. Thus $f(x_1) = 0_B = f(0_A)$, so $\mu(x_1) = \mu(0_A)$ because μ is f-invariant. Hence, $\mu(0_A) = \beta(0_B)$. Let $y \in B$. Since f is surjective, we have $f^{-1}(y) \neq \emptyset$. Then there exists $x \in f^{-1}(y)$ such that $\mu(x) = \beta(y)$, so

$$\begin{aligned}\max\{\beta(0_B), \varepsilon\} &= \max\{\mu(0_A), \varepsilon\}\\ &\geq \min\{\mu(x), \delta\}\\ &= \min\{\beta(y), \delta\}.\end{aligned}$$

Next, let $a, b, c \in B$. By Lemma 5.3, there exist $a_0 \in f^{-1}(a), b_0 \in f^{-1}(b)$, and $c_0 \in f^{-1}(c)$ such that $\beta(b) = \mu(b_0), \beta(a * c) = \mu(a_0 \cdot c_0)$, and $\beta(a * (b * c)) = \mu(a_0 \cdot (b_0 \cdot c_0))$. Thus

$$\begin{aligned}\max\{\beta(a * c), \varepsilon\} &= \max\{\mu(a_0 \cdot c_0), \varepsilon\} \\ &\geq \min\{\mu(a_0 \cdot (b_0 \cdot c_0)), \mu(b_0), \delta\} \\ &= \min\{\beta(a * (b * c)), \beta(b), \delta\}.\end{aligned}$$

Hence, β is a fuzzy UP-ideal with thresholds ε and δ of B.

(4) Assume that μ is an f-invariant fuzzy strongly UP-ideal with thresholds ε and δ of A with sup property. Since $f(0_A) = 0_B$, we have $f^{-1}(0_B) \neq \emptyset$. Then there exists $x_1 \in f^{-1}(0_B)$ such that $\mu(x_1) = \beta(0_B)$. Thus $f(x_1) = 0_B = f(0_A)$, so $\mu(x_1) = \mu(0_A)$ because μ is f-invariant. Hence, $\mu(0_A) = \beta(0_B)$. Let $y \in B$. Since f is surjective, we have $f^{-1}(y) \neq \emptyset$. Then there exists $x \in f^{-1}(y)$ such that $\mu(x) = \beta(y)$, so

$$\begin{aligned}\max\{\beta(0_B), \varepsilon\} &= \max\{\mu(0_A), \varepsilon\} \\ &\geq \min\{\mu(x), \delta\} \\ &= \min\{\beta(y), \delta\}.\end{aligned}$$

Next, let $a, b, c \in B$. By Lemma 5.3, there exist $a_0 \in f^{-1}(a), b_0 \in f^{-1}(b)$, and $c_0 \in f^{-1}(c)$ such that $\mu(a_0) = \beta(a), \mu(b_0) = \beta(b)$, and $\mu((c_0 \cdot b_0) \cdot (c_0 \cdot a_0)) = \beta((c \cdot b) \cdot (c \cdot a))$. Thus

$$\begin{aligned}\max\{\beta(a), \varepsilon\} &= \max\{\mu(a_0), \varepsilon\} \\ &\geq \min\{\mu((c_0 \cdot b_0) \cdot (c_0 \cdot a_0)), \mu(b_0), \delta\} \\ &= \min\{\beta((c \cdot b) \cdot (c \cdot a)), \beta(b), \delta\}.\end{aligned}$$

Hence, β is a fuzzy strongly UP-ideal with thresholds ε and δ of B. □

Theorem 5.5. *Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras and let $f \colon A \to B$ be a UP-homomorphism. Then the following statements hold:*

(1) if β is a fuzzy UP-subalgebra with thresholds ε and δ of B, then μ is a fuzzy UP-subalgebra with thresholds ε and δ of A,

(2) if β is a fuzzy UP-filter with thresholds ε and δ of B, then μ is a fuzzy UP-filter with thresholds ε and δ of A,

(3) if β is a fuzzy UP-ideal with thresholds ε and δ of B, then μ is a fuzzy UP-ideal with thresholds ε and δ of A, and

(4) if β is a fuzzy strongly UP-ideal with thresholds ε and δ of B, then μ is a fuzzy strongly UP-ideal with thresholds ε and δ of A.

Proof. (1) Assume that β is a fuzzy UP-subalgebra with thresholds ε and δ of B. Let $x, y \in A$. Then

$$\begin{aligned}\max\{\mu(x \cdot y), \varepsilon\} &= \max\{(\beta \circ f)(x \cdot y), \varepsilon\} \\ &= \max\{\beta(f(x \cdot y)), \varepsilon\} \\ &= \max\{\beta(f(x) * f(y)), \varepsilon\} \\ &\geq \min\{\beta(f(x)), \beta(f(y)), \delta\} \\ &= \min\{(\beta \circ f)(x), (\beta \circ f)(y), \delta\} \\ &= \min\{\mu(x), \mu(y), \delta\}.\end{aligned}$$

Hence, μ is a fuzzy UP-subalgebra with thresholds ε and δ of A.

(2) Assume that β is a fuzzy UP-filter with thresholds ε and δ of B. Let $x \in A$. Then

$$\begin{aligned} \max\{\mu(0_A), \varepsilon\} &= \max\{(\beta \circ f)(0_A), \varepsilon\} \\ &= \max\{\beta(f(0_A)), \varepsilon\} \\ &= \max\{\beta(0_B), \varepsilon\} && (f(0_A) = 0_B) \\ &\geq \min\{\beta(f(x)), \delta\} \\ &= \min\{(\beta \circ f)(x), \delta\} \\ &= \min\{\mu(x), \delta\}. \end{aligned}$$

Let $x, y \in A$. Then

$$\begin{aligned} \max\{\mu(y), \varepsilon\} &= \max\{(\beta \circ f)(y), \varepsilon\} \\ &= \max\{\beta(f(y)), \varepsilon\} \\ &\geq \min\{\beta(f(x) * f(y)), \beta(f(x)), \delta\} \\ &= \min\{\beta(f(x \cdot y)), \beta(f(x)), \delta\} \\ &= \min\{(\beta \circ f)(x \cdot y), (\beta \circ f)(x), \delta\} \\ &= \min\{\mu(x \cdot y), \mu(x), \delta\}. \end{aligned}$$

Hence, μ is a fuzzy UP-filter with thresholds ε and δ of A.

(3) Assume that β is a fuzzy UP-ideal with thresholds ε and δ of B. Let $x \in A$. Then

$$\begin{aligned} \max\{\mu(0_A), \varepsilon\} &= \max\{(\beta \circ f)(0_A), \varepsilon\} \\ &= \max\{\beta(f(0_A)), \varepsilon\} \\ &= \max\{\beta(0_B), \varepsilon\} && (f(0_A) = 0_B) \\ &\geq \min\{\beta(f(x)), \delta\} \\ &= \min\{(\beta \circ f)(x), \delta\} \\ &= \min\{\mu(x), \delta\}. \end{aligned}$$

Let $x, y, z \in A$. Then

$$\begin{aligned} \max\{\mu(x \cdot z), \varepsilon\} &= \max\{(\beta \circ f)(x \cdot z), \varepsilon\} \\ &= \max\{\beta(f(x \cdot z)), \varepsilon\} \\ &= \min\{\beta(f(x) * f(z)), \delta\} \\ &\geq \min\{\beta(f(x) * (f(y) * f(z))), \beta(f(y)), \delta\} \\ &= \min\{\beta(f(x) * f(y \cdot z)), \beta(f(y)), \delta\} \\ &= \min\{\beta(f(x \cdot (y \cdot z))), \beta(f(y)), \delta\} \\ &= \min\{(\beta \circ f)(x \cdot (y \cdot z)), (\beta \circ f)(y), \delta\} \\ &= \min\{\mu(x \cdot (y \cdot z)), \mu(y), \delta\}. \end{aligned}$$

Hence, μ is a fuzzy UP-ideal with thresholds ε and δ of A.

(4) Assume that β is a fuzzy strongly UP-ideal with thresholds ε and δ of B. Let $x \in A$.

Then

$$\begin{aligned}\max\{\mu(0_A),\varepsilon\} &= \max\{(\beta\circ f)(0_A),\varepsilon\}\\ &= \max\{\beta(f(0_A)),\varepsilon\}\\ &= \max\{\beta(0_B),\varepsilon\} && (f(0_A)=0_B)\\ &\geq \min\{\beta(f(x)),\delta\}\\ &= \min\{(\beta\circ f)(x),\delta\}\\ &= \min\{\mu(x),\delta\}.\end{aligned}$$

Let $x, y, z \in A$. Then

$$\begin{aligned}\max\{\mu(x),\varepsilon\} &= \max\{(\beta\circ f)(x),\varepsilon\}\\ &= \max\{\beta(f(x)),\varepsilon\}\\ &\geq \min\{\beta((f(z)*f(y))*(f(z)*f(x))),\beta(f(y)),\delta\}\\ &= \min\{\beta(f(z\cdot y)*f(z\cdot x)),\beta(f(y)),\delta\}\\ &= \min\{\beta(f((z\cdot y)\cdot(z\cdot x)),\beta(f(y)),\delta\}\\ &= \min\{(\beta\circ f)((z\cdot y)\cdot(z\cdot x)),(\beta\circ f)(y),\delta\}\\ &= \min\{\mu((z\cdot y)\cdot(z\cdot x)),\mu(y),\delta\}.\end{aligned}$$

Hence, μ is a fuzzy strongly UP-ideal with thresholds ε and δ of A. □

Definition 5.6. Let f be a function from a nonempty set X to a nonempty set Y. If μ is a fuzzy set in X, then the fuzzy set β in Y defined by

$$\beta(y) = \begin{cases} \inf\{\mu(t)\}_{t\in f^{-1}(y)} & \text{if } f^{-1}(y)\neq\emptyset,\\ \\ 1 & \text{if otherwise}\end{cases}$$

is said to be the *image of μ under f*. Similarly, if β is a fuzzy set in Y, then the fuzzy set $\mu = \beta\circ f$ in X (i.e., the fuzzy set defined by $\mu(x) = \beta(f(x))$ for all $x \in X$) is called the *preimage of β under f*.

Definition 5.7. [10] A fuzzy set f in A has inf *property* if for any nonempty subset T of A, there exists $t_0 \in T$ such that $f(t_0) = \inf\{f(t)\}_{t\in T}$.

Lemma 5.8. *[6] Let $(A,\cdot,0_A)$ and $(B,*,0_B)$ be UP-algebras and let $f\colon A\to B$ be a UP-epimorphism. Let μ be an f-invariant fuzzy set in A with* inf *property. For any $a, b \in B$, there exist $a_0 \in f^{-1}(a)$ and $b_0 \in f^{-1}(b)$ such that $\beta(a) = \mu(a_0), \beta(b) = \mu(b_0)$, and $\beta(a*b) = \mu(a_0\cdot b_0)$.*

Theorem 5.9. *Let $(A,\cdot,0_A)$ and $(B,*,0_B)$ be UP-algebras and let $f\colon A\to B$ be a UP-epimorphism. Then the following statements hold:*

(1) if μ is an f-invariant fuzzy UP-subalgebra with thresholds ε and δ of A with inf *property, then β is a fuzzy UP-subalgebra with thresholds ε and δ of B,*

(2) if μ is an f-invariant fuzzy UP-filter with thresholds ε and δ of A with inf *property, then β is a fuzzy UP-filter with thresholds ε and δ of B,*

(3) if μ is an f-invariant fuzzy UP-ideal with thresholds ε and δ of A with inf *property, then β is a fuzzy UP-ideal with thresholds ε and δ of B, and*

(4) if μ is an f-invariant fuzzy strongly UP-ideal with thresholds ε and δ of A with inf *property, then β is a fuzzy strongly UP-ideal with thresholds ε and δ of B.*

Proof. (1) Assume that μ is an f-invariant fuzzy UP-subalgebra with thresholds ε and δ of A with inf property. Let $a, b \in B$. By Lemma 5.8, there exist $a_0 \in f^{-1}(a)$ and $b_0 \in f^{-1}(b)$ such that $\beta(a) = \mu(a_0), \beta(b) = \mu(b_0)$, and $\beta(a * b) = \mu(a_0 \cdot b_0)$. Thus

$$\begin{aligned}\max\{\beta(a * b), \varepsilon\} &= \max\{\mu(a_0 \cdot b_0), \varepsilon\} \\ &\geq \min\{\mu(a_0), \mu(b_0), \delta\} \\ &= \min\{\beta(a), \beta(b), \delta\}.\end{aligned}$$

Hence, β is a fuzzy UP-subalgebra with thresholds ε and δ of B.

(2) Assume that μ is an f-invariant fuzzy UP-filter with thresholds ε and δ of A with inf property. Since $f(0_A) = 0_B$, we have $f^{-1}(0_B) \neq \emptyset$. Then there exist $x_1 \in f^{-1}(0_B)$ such that $\mu(x_1) = \beta(0_B)$. Thus $f(x_1) = 0_B = f(0_A)$, so $\mu(x_1) = \mu(0_A)$ because μ is f-invariant. Hence, $\mu(0_A) = \beta(0_B)$. Let $y \in B$. Since f is surjective, we have $f^{-1}(y) \neq \emptyset$. Then there exist $x \in f^{-1}(y)$ such that $\mu(x) = \beta(y)$, so

$$\begin{aligned}\max\{\beta(0_B), \varepsilon\} &= \max\{\mu(0_A), \varepsilon\} \\ &\geq \min\{\mu(x), \delta\} \\ &= \min\{\beta(y), \delta\}.\end{aligned}$$

Next, let $a, b \in B$. By Lemma 5.8, there exist $a_0 \in f^{-1}(a)$ and $b_0 \in f^{-1}(b)$ such that $\beta(a) = \mu(a_0), \beta(b) = \mu(b_0)$, and $\beta(a * b) = \mu(a_0 \cdot b_0)$. Thus

$$\begin{aligned}\max\{\beta(b), \varepsilon\} &= \max\{\mu(b_0), \varepsilon\} \\ &\geq \min\{\mu(a_0 \cdot b_0), \mu(a_0), \delta\} \\ &= \min\{\beta(a * b), \beta(a), \delta\}.\end{aligned}$$

Hence, β is a fuzzy UP-filter with thresholds ε and δ of B.

(3) Assume that μ is an f-invariant fuzzy UP-ideal with thresholds ε and δ of A with inf property. Since $f(0_A) = 0_B$, we have $f^{-1}(0_B) \neq \emptyset$. Then there exist $x_1 \in f^{-1}(0_B)$ such that $\mu(x_1) = \beta(0_B)$. Thus $f(x_1) = 0_B = f(0_A)$, so $\mu(x_1) = \mu(0_A)$ because μ is f-invariant. Hence, $\mu(0_A) = \beta(0_B)$. Let $y \in B$. Since f is surjective, we have $f^{-1}(y) \neq \emptyset$. Then there exist $x \in f^{-1}(y)$ such that $\mu(x) = \beta(y)$, so

$$\begin{aligned}\max\{\beta(0_B), \varepsilon\} &= \max\{\mu(0_A), \varepsilon\} \\ &\geq \min\{\mu(x), \delta\} \\ &= \min\{\beta(y), \delta\}.\end{aligned}$$

Next, let $a, b, c \in B$. By Lemma 5.8, there exist $a_0 \in f^{-1}(a), b_0 \in f^{-1}(b)$, and $c_0 \in f^{-1}(c)$ such that $\beta(b) = \mu(b_0), \beta(a * c) = \mu(a_0 \cdot c_0)$, and $\beta(a * (b * c)) = \mu(a_0 \cdot (b_0 \cdot c_0))$. Thus

$$\begin{aligned}\max\{\beta(a * c), \varepsilon\} &= \max\{\mu(a_0 \cdot c_0), \varepsilon\} \\ &\geq \min\{\mu(a_0 \cdot (b_0 \cdot c_0)), \mu(b_0), \delta\} \\ &= \min\{\beta(a * (b * c)), \beta(b), \delta\}.\end{aligned}$$

Hence, β is a fuzzy UP-ideal with thresholds ε and δ of B.

(4) Assume that μ is an f-invariant fuzzy strongly UP-ideal with thresholds ε and δ of A with inf property. Since $f(0_A) = 0_B$, we have $f^{-1}(0_B) \neq \emptyset$. Then there exist $x_1 \in f^{-1}(0_B)$

such that $\mu(x_1) = \beta(0_B)$. Thus $f(x_1) = 0_B = f(0_A)$, so $\mu(x_1) = \mu(0_A)$ because μ is f-invariant. Hence, $\mu(0_A) = \beta(0_B)$. Let $y \in B$. Since f is surjective, we have $f^{-1}(y) \neq \emptyset$. Then there exist $x \in f^{-1}(y)$ such that $\mu(x) = \beta(y)$, so

$$\begin{aligned} \max\{\beta(0_B), \varepsilon\} &= \max\{\mu(0_A), \varepsilon\} \\ &\geq \min\{\mu(x), \delta\} \\ &= \min\{\beta(y), \delta\}. \end{aligned}$$

Next, let $a, b, c \in B$. By Lemma 5.8, there exist $a_0 \in f^{-1}(a), b_0 \in f^{-1}(b)$, and $c_0 \in f^{-1}(c)$ such that $\beta(a) = \mu(a_0), \beta(b) = \mu(b_0)$, and $\beta((c * b) * (c * a)) = \mu((c_0 \cdot b_0) \cdot (c_0 \cdot a_0))$. Thus

$$\begin{aligned} \max\{\beta(a), \varepsilon\} &= \max\{\mu(a_0), \varepsilon\} \\ &\geq \min\{\mu((c_0 \cdot b_0) \cdot (c_0 \cdot a_0)), \mu(b_0), \delta\} \\ &= \min\{\beta((c * b) * (c * a)), \beta(b), \delta\}. \end{aligned}$$

Hence, β is a fuzzy strongly UP-ideal with thresholds ε and δ of B. □

Theorem 5.10. *Let $(A; \cdot, 0_A)$ and $(B; *, 0_B)$ be UP-algebras and let $f\colon A \to B$ be a UP-homomorphism. Then the following statements hold:*

(1) if β is a fuzzy UP-subalgebra with thresholds ε and δ of B, then μ is a fuzzy UP-subalgebra with thresholds ε and δ of A,

(2) if β is a fuzzy UP-filter with thresholds ε and δ of B, then μ is a fuzzy UP-filter with thresholds ε and δ of A,

(3) if β is a fuzzy UP-ideal with thresholds ε and δ of B, then μ is a fuzzy UP-ideal with thresholds ε and δ of A, and

(4) if β is a fuzzy strongly UP-ideal with thresholds ε and δ of B, then μ is a fuzzy strongly UP-ideal with thresholds ε and δ of A.

Proof. (1) Assume that β is a fuzzy UP-subalgebra with thresholds ε and δ of B. Let $x, y \in A$. Then

$$\begin{aligned} \max\{\mu(x \cdot y), \varepsilon\} &= \max\{(\beta \circ f)(x \cdot y), \varepsilon\} \\ &= \max\{\beta(f(x \cdot y)), \varepsilon\} \\ &= \max\{\beta(f(x) * f(y)), \varepsilon\} \\ &\geq \min\{\beta(f(x)), \beta(f(y)), \delta\} \\ &= \min\{(\beta \circ f)(x), (\beta \circ f)(y), \delta\} \\ &= \min\{\mu(x), \mu(y), \delta\}. \end{aligned}$$

Hence, μ is a fuzzy UP-subalgebra with thresholds ε and δ of A.

(2) Assume that β is a fuzzy UP-filer with thresholds ε and δ of B. Let $x \in A$. Then

$$\begin{aligned} \max\{\mu(0_A), \varepsilon\} &= \max\{(\beta \circ f)(0_A), \varepsilon\} \\ &= \max\{\beta(f(0_A)), \varepsilon\} \\ &= \max\{\beta(0_B), \varepsilon\} && (f(0_A) = 0_B) \\ &\geq \min\{\beta(f(x)), \delta\} \\ &= \min\{(\beta \circ f)(x), \delta\} \\ &= \min\{\mu(x), \delta\}. \end{aligned}$$

Let $x, y \in A$. Then

$$\begin{aligned}\max\{\mu(y), \varepsilon\} &= \max\{(\beta \circ f)(y), \varepsilon\} \\ &= \max\{\beta(f(y)), \varepsilon\} \\ &\geq \min\{\beta(f(x) * f(y)), \beta(f(x)), \delta\} \\ &= \min\{\beta(f(x \cdot y)), \beta(f(x)), \delta\} \\ &= \min\{(\beta \circ f)(x \cdot y), (\beta \circ f)(x), \delta\} \\ &= \min\{\mu(x \cdot y), \mu(x), \delta\}.\end{aligned}$$

Hence, μ is a fuzzy UP-filter with thresholds ε and δ of A.

(3) Assume that β is a fuzzy UP-ideal with thresholds ε and δ of B. Let $x \in A$. Then

$$\begin{aligned}\max\{\mu(0_A), \varepsilon\} &= \max\{(\beta \circ f)(0_A), \varepsilon\} \\ &= \max\{\beta(f(0_A)), \varepsilon\} \\ &= \max\{\beta(0_B), \varepsilon\} && (f(0_A) = 0_B) \\ &\geq \min\{\beta(f(x)), \delta\} \\ &= \min\{(\beta \circ f)(x), \delta\} \\ &= \min\{\mu(x), \delta\}.\end{aligned}$$

Let $x, y, z \in A$. Then

$$\begin{aligned}\max\{\mu(x \cdot z), \varepsilon\} &= \max\{(\beta \circ f)(x \cdot z), \varepsilon\} \\ &= \max\{\beta(f(x \cdot z)), \varepsilon\} \\ &= \max\{\beta(f(x) * f(z)), \varepsilon\} \\ &\geq \min\{\beta(f(x) * (f(y) * f(z))), \beta(f(y)), \delta\} \\ &= \min\{\beta(f(x) * f(y \cdot z)), \beta(f(y)), \delta\} \\ &= \min\{\beta(f(x \cdot (y \cdot z))), \beta(f(y)), \delta\} \\ &= \min\{(\beta \circ f)(x \cdot (y \cdot z)), (\beta \circ f)(y), \delta\} \\ &= \min\{\mu(x \cdot (y \cdot z)), \mu(y), \delta\}.\end{aligned}$$

Hence, μ is a fuzzy UP-ideal with thresholds ε and δ of A.

(4) Assume that β is a fuzzy strongly UP-ideal with thresholds ε and δ of B. Let $x \in A$. Then

$$\begin{aligned}\max\{\mu(0_A), \varepsilon\} &= \max\{(\beta \circ f)(0_A), \varepsilon\} \\ &= \max\{\beta(f(0_A)), \varepsilon\} \\ &= \max\{\beta(0_B), \varepsilon\} && (f(0_A) = 0_B) \\ &\geq \min\{\beta(f(x)), \delta\} \\ &= \min\{(\beta \circ f)(x), \delta\} \\ &= \min\{\mu(x), \delta\}.\end{aligned}$$

Let $x, y, z \in A$. Then

$$\begin{aligned}
\max\{\mu(x), \varepsilon\} &= \max\{(\beta \circ f)(x), \varepsilon\} \\
&= \max\{\beta(f(x)), \varepsilon\} \\
&\geq \min\{\beta((f(z) * f(y)) * (f(z) * f(x))), \beta(f(y)), \delta\} \\
&= \min\{\beta(f(z \cdot y) * f(z \cdot x)), \beta(f(y)), \delta\} \\
&= \min\{\beta(f((z \cdot y) \cdot (z \cdot x))), \beta(f(y)), \delta\} \\
&= \min\{(\beta \circ f)((z \cdot y) \cdot (z \cdot x)), (\beta \circ f)(y), \delta\} \\
&= \min\{\mu((z \cdot y) \cdot (z \cdot x)), \mu(y), \delta\}.
\end{aligned}$$

Hence, μ is a fuzzy strongly UP-ideal with thresholds ε and δ of A. □

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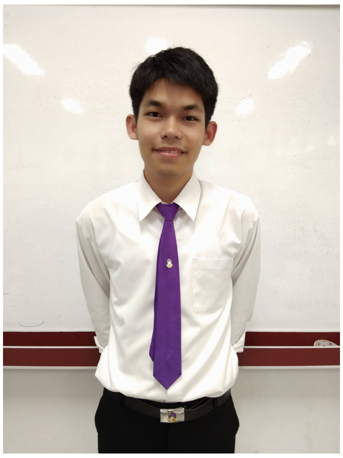

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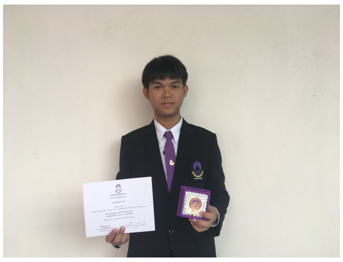

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