CONVERGENCE OF PROJECTION TYPE ITERATIVE PROCESSES OF MIXED ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

RUETHAITA YEAMPUN
WILIPHORN WITIOON
INTIRA KAPANG

An Independent Study Submitted in Partial Fulfillment of the Requirements for the Degree of Bachelor of Science Program in Mathematics

December 2016

University of Phayao

CONVERGENCE OF PROJECTION TYPE ITERATIVE PROCESSES OF MIXED ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

RUETHAITA YEAMPUN
WILIPHORN WITIOON
INTIRA KAPANG

An Independent Study Submitted in Partial Fulfillment of the Requirements for the Degree of Bachelor of Science Program in Mathematics

December 2016

University of Phayao

Advisor and Dean of School of Science have considered the independent study entitled "Convergence of projection type iterative processes of mixed asymptotically nonexpansive mappings" submitted in partial fulfillment of the requirements for the degree of Bachelor of Science Program in Mathematics is hereby approved.

....

(Dr. Uamporn Witthayarat)

Chairman

....

(Dr. Watcharaporn Cholamjiak)

Committee

....

(Assist. Prof. Dr. Tanakit Thianwan)

Committee and Advisor

....

(Assoc. Prof. Preeyanan Sanpote)

Dean of School of Science

December 2016

ACKNOWLEDGEMENTS

This independent study could not successfully completed without the Kindness of advisor, Assistant Professor Dr. Tanakit Thianwan, who gave good advice and be quidance of this independent study since start until successful. Including, he gave appreciate suggestion, encourage, checked and corrected the fault of this independent study. So, I would like to express my sincere thanks to my independent study advisor.

In addition, we are grateful for the teachers of mathematics for suggestions and all their help.

Finally, my graduation would not be achieved without best wish from my parents, who help we for everything and always gives we greatest love, will power and financial support until this study completion. And the last gratefully special thanks to my relation and my friends for their help and encouragement.

Ruethaita Yeampun
Wiliphron Witoon
Intira Kapang

ชื่อเรื่อง	การลู่เข้าของวิธีทำซ้ำแบบภาพฉายของการส่งไม่ขยาย แบบเชิงเส้นกำกับผสม
ผู้ศึกษาค้นคว้า	นางสาวฤทัยตา แย้มปั้น นางสาววิไลพร วิฑุรย์ นางสาวอินทิรา คาแพง
อาจารย์ที่ปรึกษา	ผู้ช่วยศาสตราจารย์ ดร.ธนกฤต เทียนหวาน
วิทยาศาสตรบัณฑิต	สาขาวิชาคณิตศาสตร์
คำสำคัญ	การส่งไม่ขยายแบบเชิงเส้นกำกับผสม การลู่เข้าอย่างเข้ม การลู่เข้าอย่างอ่อน จุดตรึงร่วม ปริภูมิบานาคนูนเอกรูป

บทคัดย่อ

ในงานวิจัยนี้ ได้แนะนำระเบียบวิธีการทำซ้ำแบบภาพฉายชนิดผสมของสองการส่งในตัวไม่ขยายแบบเชิงเส้นกำกับ และสองการส่งนอกตัวไม่ขยายแบบเชิงเส้นกำกับในปริภูมิบานาคนูนเอกรูป และให้ทฤษฎีบทการลู่เข้าอย่างอ่อนและอย่างเข้มในปริภูมิบานาคนูนเอกรูป

Title	CONVERGENCE OF PROJECTION TYPE ITERATIVE PROCESSES OF MIXED ASYMPTOTICALLY NONEXPANSIVE MAPPINGS
Author	Ruethaita Yeampun Wiliphorn Witoon Intira Kapang
Advisor	Dr. Tanakit Thianwan
Bachelor of Science	Program in Mathematics
Keywords	mixed asymptotically nonexpansive mapping, strong and weak convergence, common fixed point, uniformly convex Banach space

ABSTRACT

In this research, we introduce a projection type iterative scheme of mixed type for two asymptotically nonexpansive self-mappings and two asymptotically nonexpansive nonself-mappings in uniformly convex Banach spaces. Weak and Strong convergence theorems are established in uniformly convex Banach spaces.

LIST OF CONTENTS

Page

CHAPTER 1
Introduction

Let K be a nonempty closed convex subset of a real normed linear space E. A self-mapping $T : K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. A self-mapping $T : K \to K$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{1.1}$$

for all $x, y \in K$ and $n \geq 1$.

A mapping $T : K \to K$ is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\| \tag{1.2}$$

for all $x, y \in K$ and $n \geq 1$.

It is easy to see that if T is an asymptotically nonexpansive, then it is uniformly L-Lipschitzian with the uniform Lipschitz constant $L = sup\{k_n : n \geq 1\}$.

Fixed-point iteration process for nonexpansive self-mappings including Mann and Ishikawa iteration processes have been studied extensively by various authors [1, 9, 11, 16, 17, 22]. For nonexpansive nonself-mappings, some authors [10, 14, 25, 27, 32] have studied the strong and weak convergence theorems in Hilbert space or uniformly convex Banach space. In 2972, Goebel and Kirk [4] introduced the class of asymptotically nonexpansive self-mappings, who proved that if K is nonempty closed convex subset of real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping on C, then T has a fixed point.

In 1991, Schu [23] introduced a modified Mann iteration process to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbret space. More precisely, he proved the following theorem.

Theorem 1.1 (see [23]). Let H be a Hilbert space, and let K be a nonempty closed convex and bounded subset of H. Let $T : K \to K$ be an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$ for all $n \geq 1$, $lim_{n\to\infty} k_n = 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}$ be a sequence in $[0.1]$ satisfying the condition $0 < a \leq \alpha_n \leq b < 1, n \geq 1$, for some constant a, b. Then the sequence $\{x_n\}$ generated from an arbitrary $x_1 \in K$ by the relation

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \qquad n \geq 1, \tag{1.3}$$

converges strongly to some fixed point of T.

Since then, Schu's iteration process has been widely used to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert or Banach spaces (see [16],[20],[21],[23],[28]).

The concept of asymptotically nonexpansive nonself-mappings was introduced by Chidume, Ofoedu, and Zegeye [2] in 2003 as the generlization of asymptotically nonexpansive self-mappings. The nonself of asymptotically nonexpansive nonself-mapping is defined as follows.

Definition 1.2 (see [2]). Let K be a nonempty subset of a real normed linear space E. Let $P : E \to K$ be a nonexpansive retraction of E onto K. A nonself-mapping $T : K \to E$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ as $n \to \infty$ such that

$$\|T(PT)^{n-1}x - T(PT)_1^{n-1}y\| \leq k_n\|x - y\| \tag{1.4}$$

for all $x, y \in K$ and $n \geq 1$. A non-self-mapping T is said to be uniformly $L - Lipschitzian$ if there exists a constant $L \geq 0$ such that

$$\|T(PT)^{n-1}x - T(PT)_1^{n-1}y\| \leq L\|x - y\| \tag{1.5}$$

for all $x, y \in K$ and $n \geq 1$.

We denote by $(PT)^0$ the identity map from K onto itself. In [2], the authors studied the following iterative sequence: $x_1 \in K$,

$$x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \tag{1.6}$$

to approximate some fixed point of T under suitable conditions.

If T is a self-mapping, then P becomes the identity mapping so that (1.4) and (1.5) reduce to (1.1) and (1.2), respectively, and (1.6) reduces to (1.3).

In 2006, Wang[31] generalized the iteration process (1.8) as follows: $x_1 \in K$,

$$y_n = P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n),$$
$$x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n),\ n \geq 1, \tag{1.7}$$

where $T_1, T_2 : K \to E$ are asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in [0,1). He proved strong and weak convergence of the sequence $\{\alpha_n\}$ defined by (1.7) to a common fixed point of T_1 and T_2 under appropriate conditions. Meanwhile, the results of [31] generalized the results of [2].

In 2009, a new iterative scheme which is called the projection type Ishikawa iteration for two asymptotically nonexpansive nonself-mappings was defined and constructed by Thianwan [30]. It is given as follows:

$$y_n = P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n),$$
$$x_{n+1} = P((1-\alpha_n)y_n + \alpha_n T_1(PT_1)^{n-1}y_n),\ n \geq 1, \tag{1.8}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate real sequences in [0,1). He studied the scheme for two asymptotically nonexpansive nonself-mappings and proved strong and weak convergence of the sequences $\{x_n\}$ and $\{y_n\}$ to a common fixed point of T_1, T_2 under suitable conditions in a uniformly convex Banach space.

Note that Thianwan process (1.8) and Wang process (1.7) are independent neither reduces to the other.

If $T_1 = T_2$ and $\beta_n = 0$ for all $n \geq 1$, then (1.8) reduces to (1.6). It also can be reduces to Schu process (1.3).

Recently, Guo, Cho and Guo [7] studied the following iteration scheme: $x_1 \in K$,

$$y_n = P((1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1}x_n),$$
$$x_{n+1} = P((1-\alpha_n)S_1^n x_n + \alpha_n T_1(PT_1)^{n-1}y_n),\ n \geq 1, \tag{1.9}$$

where $S_1, S_2 : K \to K$ are asymptotically nonexpansive self-mappings, $T_1, T_2 : K \to E$ are asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in [0,1). They studied the strong and weak convergence of the iterative scheme (1.9) under proper conditions.

If S_1 and S_2 are the identity mappings, then the iterative scheme (1.9) reduces to the scheme (1.7).

Motivated by these recent works, we introduce and study a new iterative scheme in this paper. The scheme is defind as follows.

Let E be a real Banach space, K be a nonempty closed convex subset of E and $P : E \to K$ be a nonexpansive retraction of E onto K. Let $S_1, S_2 : K \to K$ be two asymptotically nonexpansive self-mappings and $T_1, T_2 : K \to E$ be two asymptotically nonexpansive nonself-mappings. Then, we define the new iteration scheme of mixed type as follows : $x_1 \in K$,

$$y_n = P((1-\beta_n)S_2^n x_n + \beta_n T_2(PT)^{n-1}x_n),$$
$$x_{n+1} = P((1-\alpha_n)S_1^n y_n + \alpha_n T_1(PT)^{n-1}y_n), \qquad n \geq 1, \tag{1.10}$$

where $\{\alpha_n\}, \{\beta_n\}$ are two sequences [0,1).

The iterative scheme (1.10) is called the projective type iterative process for mixed type of asymptotically nonexpansive mappings. If S_1 and S_2 are the identity mappings, then the iterative scheme (1.10) reduces to (1.8).

Note that (1.9) and (1.10) are independent neither reduces to the other.

The purpose of this paper is to construct an iteration scheme for approximating common fixed points of two asymptotically nonexpansive self-mappings and two asymptotically nonexpansive nonself-mappings and to prove some strong and weak convergence theorems for such mappings in a real uniformly convex Banach space.

CHAPTER 2
Preliminaries

We denote the set of common fixed points of S_1, S_2, T_1 and T_2 by $F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$ and denote the distance between a point z and a set A in E by $d(z, A) = \inf_{x \in A} \|z - x\|$.

Now, we recall some well-known concepts and results.

Let E be a real Banach space, E^* be the dual space of E and $J : E \to 2^{E^*}$ be the $normalized\ duality\ mapping$ defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|\|f\|, \|f\| = \|x\|\}$$

for all $x \in E$, where $\langle \cdot, \cdot \rangle$ denotes duality pairing between E and E^*. A single-valued normalized duality mapping is denoted by j.

A subset K of a real Banach space E is called a $retract$ of E [2] if there exists a continuous mapping $P : E \to K$ such that $Px = x$ for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : E \to E$ is called a $retraction$ if $P^2 = P$. It follows that if a mapping P is a retraction, than $Py = y$ for all y in the range of P.

Recall that a Banach space E is said to satisfy $Opial's\ condition$ [15] if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ implying that

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|.$$

A mapping $T : K \to E$ is said to be semi-compact if, for any sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $x^* \in K$.

A Banach space E is said to have a $Fr\acute{e}chet$ *differentiable* norm [17] if, for all $x \in \mathcal{U} = \{x \in E : \|x\| = 1\}$,

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists and is attained uniformly in $y \in \mathcal{U}$.

A Banach space E is said to have the *Kadec-Klee property* [5] if for every sequence $\{x_n\}$ in $E, x_n \to x$ weakly and $\|x_n\| \to \|x\|$, if follows that $x_n \to x$ strongly.

In the sequel, the following lemmas are needed to prove our main results.

Lemma 2.1 [26] *Let* $\{a_n\}, \{b_n\}$ *and* $\{c_n\}$ *be three nonnegative sequences satisfying the following condition*:

$$a_{n+1} \leq (1 + b_n)a_n + c_n$$

for each $n \geq n_0$, where n_0 is some nonnegative integer, $\sum_{n=n_0}^{\infty} b_n < \infty$ and $\sum_{n=n_0}^{\infty} c_n < \infty$. Then $lim_{n\to\infty} a_n$ exists.

Lemma 2.2 [23] *Let* E *be a real uniformly convex Banach space and* $0 < p \leq t_n < q < 1$ *for each* $n \geq 1$. *Also, suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are two sequences of such that*

$$\limsup_{n\to\infty} \|x_n\| \leq r, \quad \limsup_{n\to\infty} \|y_n\| \leq r, \quad \lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = r$$

hold for some $r \geq 0$. Then $lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.3 [2] *Let E be a real uniformly convex Banach space,* K *be a nonempty closed convex subset of* E *and* $T : K \to E$ *be an asymptotically nonexpansive mapping with a sequence* $\{k_n\} \subset [1, \infty)$ *and* $k_n \to 1$ *as* $n \to \infty$. *Then* $I - T$ *is demiclosed at zero, i.e., if* $x_n \to x$ *weakly and* $x_n - Tx_n \to 0$ *strongly, then* $x \in F(T)$, *where* $F(T)$ *is the set of fixed points of T.*

Lemma 2.4 [3] *Let E be a uniformly convex Banach space and K be a convex subset of E. Then there exists a strictly increasing continuous convex function* $\gamma : [0, \infty) \to [0, \infty)$ *with* $\gamma(0) = 0$ *such that, for each mapping* $S : K \to K$ *with a Lipschitz constant* $L > 0$,

$$\|\alpha_n + (1-\alpha)Sy - S(\alpha x + (1-\alpha)y)\| \leq L\gamma^1(\|x-y\| - \tfrac{1}{L}\|Sx - Sy\|)$$

for all $x, y \in K$ and $0 < \alpha < 1$.

Lemma 2.5 [3] *Let E be a uniformly convex Banach space such that its dual space E^* has the Kadec-Klee property. Suppose $\{x_n\}$ is a bounded sequence and $f_1, f_2 \in W_w(\{x_n\})$ such that*

$$\lim_{n\to\infty} \|\alpha x_n + (1-\alpha)f_1 - f_2\|$$

exists for all $\alpha \in [0,1]$, where $W_w(\{x_n\})$ denotes the set of all weak subsequential limits of $\{x_n\}$. Then $f_1 = f_2$.

CHAPTER 3
Main Results

In this chapter, we prove theorems of strong and weak convergence of the iterative scheme given in (1.10) to a common fixed point of mixed type of two asymptotically nonexpansive self-mappings and two asymptotically nonexpansive nonself-mappings in uniformly convex Banach spaces.

In order to prove our main results, the following lemmas are needed.

Lemma 3.1 Let E be a real uniformly convex Banach space and K a nonempty closed convex nonexpansive retract of E with P as a nonexpansive retraction. Let S_1, S_2 : $K \longrightarrow K$ be two asymptotically nonexpansive self-mappings with$\{k_n^{(1)}\}$, $\{k_n^{(2)}\} \subset [1, \infty)$ and T_1, $T_2 : K \to E$ be two asymptotically nonexpansive nonself-mappings with $\{l_n^{(1)}\}$, $\{l_n^{(2)}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)} - 1) < \infty$ for $i = 1$, 2, respectively and $F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1)$. From an arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ using (1.10) Then

(1) $\lim_{n\to\infty} \|x_n - q\|$ exists for any $q \in F$;

(2) $\lim_{n\to\infty} d(x_n, F)$ exists.

Proof Let $q \in F$. Setting $h_n = max\{k_n^{(1)},\ k_n^{(2)},\ l_n^{(1)},\ l_n^{(2)}\}$. Using (1.10), we have

$$\begin{aligned}
\|y_n - q\| &= \|P((1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1}x_n) - q\| \\
&= \|P((1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1}x_n) - P(q)\| \\
&\leq \|(1-\beta_n)(S_2^n x_n - q) + \beta_n(T_2(PT_2)^{n-1}x_n - q)\| \\
&\leq (1-\beta_n)h_n\|x_n - q\| + \beta_n h_n\|x_n - q\| \\
&= h_n\|x_n - q\|, \qquad (3.1)
\end{aligned}$$

and so

$$\begin{aligned}
\|x_{n+1}-q\| &= \|P((1-\alpha_n)S_1^n y_n+\alpha_n T_1(PT_1)^{n-1}y_n)-q\| \\
&= \|P((1-\alpha_n)S_1^n y_n+\alpha_n T_1(PT_1)^{n-1}y_n)-P(q)\| \\
&\leq \|(1-\alpha_n)(S_1^n y_n-q)+\alpha_n(T_1(PT_1)^{n-1}y_n-q)\| \\
&\leq (1-\alpha_n)h_n\|y_n-q\|+\alpha_n h_n\|y_n-q\| \\
&= h_n\|y_n-q\| \\
&\leq h_n^2\|x_n-q\| \\
&= (1+(h_n^2-1))\|x_n-q\|. \qquad (3.2)
\end{aligned}$$

Since $\sum_{n=1}^{\infty}(k_n^{(i)}-1)<\infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)}-1)<\infty$ for $i=1,2$, we have $\sum_{n=1}^{\infty}(h_n^2-1)<\infty$. It follows from Lemma 2.1 that $\lim_{n\to\infty}\|x_n-q\|$ exists.

(2) Taking the infimum over all $q\in F$ in (3.2), we have

$$d(x_{n+1},F)\leq(1+(h_n^2-1))d(x_n,F)$$

for each $n\geq 1$. It follows from $\sum_{n=1}^{\infty}(h_n^2-1)<\infty$ and Lemma 2.1 that the conclusion (2) holds. This completes the proof.

Lemma 3.2 Let E be a real uniformly convex Banach space and K a nonempty closed convex nonexpansive retract of E with P as a nonexpansive retraction. Let $S_1,S_2:K\to K$ be two asymptotically nonexpansive self-mappings with $\{k_n^{(1)}\},\{k_n^{(2)}\}\subset[1,\infty)$ and $T_1,T_2:K\to E$ be two asymptotically nonexpansive nonself-mappings with $\{l_n^{(1)}\},\{l_n^{(2)}\}\subset[1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)}-1)<\infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)}-1)<\infty$ for $i=1,2$, respectively and $F=F(S_1)\cap F(S_2)\cap F(T_1)\cap F(T_2)\neq\emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon,1-\epsilon]$ for some $\epsilon\in(0,1)$. From an arbitrary $x_1\in K$, define the sequence $\{x_n\}$ using (1.10). If $\|x-T_iy\|\leq\|S_ix-T_iy\|$ for all $x,y\in K$ and $i=1,2$, then $\lim_{n\to\infty}\|x_n-S_ix_n\|=\lim_{n\to\infty}\|x_n-T_ix_n\|=0$ for $i=1,2$.

Proof Suppose that $\|x-T_iy\| \leq \|S_ix-T_iy\|$ for all $x, y \in K$ and $i = 1, 2$. Let $q \in F$. Set $h_n = max\{k_n^{(1)}, k_n^{(2)}, l_n^{(1)}, l_n^{(2)}\}$. By Lemma 3.1, we are that $lim_{n\to\infty}\|x_n-q\|$ exists. Assume that $\lim_{n\to\infty} \|x_n-q\| = c$. Since $\sum_{n=1}^{\infty}(h_n^2-1) < \infty$ and $\lim_{n\to\infty} \|x_{n+1}-q\| = c$, letting $n \to \infty$ in the inequality (3.2), we have

$$\lim_{n\to\infty} \|(1-\alpha_n)(S_1^n y_n - q) + \alpha_n T_1(PT_1)^{n-1} y_n - q)\| = c. \tag{3.3}$$

In addition, $\|S_1^n y_n - q\| \leq k_n^{(1)}\|y_n - q\|$, taking the lim sup on both sides in this inequality, we have

$$\limsup_{n\to\infty} \|S_1^n y_n - q\| \leq c. \tag{3.4}$$

Taking the lim sup on both sides in the inequality (3.1), we obtain $\limsup_{n\to\infty} \|y_n-q\| \leq c$, and so

$$\limsup_{n\to\infty} \|T_1(PT_1)^{n-1} y_n - q\| \leq \limsup_{n\to\infty} l_n^{(1)} \|y_n - q\| \leq c. \tag{3.5}$$

By using (3.3), (3.4), (3.5) and Lemma 2.2, we have

$$\lim_{n\to\infty} \|S_1^n y_n - T_1(PT_1)^{n-1} y_n\| = 0. \tag{3.6}$$

Since

$$\|y_n - T_1(PT_1)^{n-1} y_n\| \leq \|S_1^n y_n - T_1(PT_1)^{n-1} y_n\|. \tag{3.7}$$

Letting $n \to \infty$ in the inequality (3.7), by (3.6), we have

$$\lim_{n\to\infty} \|y_n - T_1(PT_1)^{n-1} y_n\| = 0. \tag{3.8}$$

From (3.2), we have

$$\|x_{n+1} - q\| \;\leq\; h_n\|y_n - q\| \leq h_n^2\|y_n - q\|. \tag{3.9}$$

Taking the lim inf on both sidies in the inequality (3.9), we have

$$\liminf_{n\to\infty} \|y_n - q\| \geq c. \tag{3.10}$$

Since $\limsup_{n\to\infty} \|y_n - q\| \leq c$, by (3.10), we have $\lim_{n\to\infty} \|y_n - q\| = c$. This implies that

$$\begin{aligned} c = \lim \|y_n - q\| &\leq \lim_{n\to\infty} \|(1-\beta_n)(S_2^n x_n - q) + \beta_n(T_2(PT_2)^{n-1}x_n - q)\| \\ &\leq \lim_{n\to\infty} \|x_n - q\| = c, \end{aligned}$$

and so

$$\lim_{n\to\infty} \|(1-\beta_n)(S_2^n x_n - q) + \beta_n(T_2(PT_2)^{n-1}x_n - q)\| = c. \tag{3.11}$$

In addition, we have

$$\limsup_{n\to\infty} \|S_2^n x_n - q\| \leq \limsup_{n\to\infty} k_n^{(2)}\|x_n - q\| = c \tag{3.12}$$

and

$$\limsup_{n\to\infty} \|T_2(PT_2)^{n-1}x_n - q\| \leq \limsup_{n\to\infty} l_n^{(2)}\|x_n - q\| = c. \tag{3.13}$$

It follows from (3.11), (3.12), (3.13) and Lemma 2.2 that

$$\lim_{n\to\infty} \|S_2^n x_n - T_2(PT_2)^{n-1}x_n\| = 0. \tag{3.14}$$

Now, we prove that

$$\lim_{n\to\infty} \|x_n - T_1x_n\| = \lim_{n\to\infty} \|x_n - T_2x_n\| = 0.$$

Indeed, since $\|x_n - T_2(PT_2)^{n-1}x_n\| \leq \|S_2^n x_n - T_2(PT_2)^{n-1}x_n\|$. (3.15)

Using (3.14) and (3.15), we have

$$\lim_{n\to\infty} \|x_n - T_2(PT_2)^{n-1}x_n\| = 0. \tag{3.16}$$

Since $S_2^n x_n = P(S_2^n x_n)$ and $P : E \to K$ is nonexpansive rectraction of E onto K, we have

$$\begin{aligned} \|y_n - S_2^n x_n\| &\leq \|(1-\beta_n)(S_2^n x_n - S_2^n x_n) + \beta_n(T_2(PT_2)^{n-1}x_n - S_2^n x_n)\| \\ &\leq \beta_n \|T_2(PT_2)^{n-1}x_n - S_2^n x_n\|. \end{aligned}$$

Using (3.14), we have

$$\lim_{n\to\infty} \|y_n - S_2^n x_n\| = 0. \tag{3.17}$$

Furthermore, we have

$$\begin{aligned} \|y_n - x_n\| \leq \|y_n - S_2^n x_n\| + \|S_2^n x_n - T_2(PT_2)^{n-1}x_n\| \\ + \|T_2(PT_2)^{n-1}x_n - x_n\|. \end{aligned} \tag{3.18}$$

It follows from (3.14), (3.16), (3.17) and (3.18) that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.19}$$

Since

$$\|x_n - T_1(PT_1)^{n-1}x_n\| \leq \|S_1^n x_n - T_1(PT_1)^{n-1}x_n\|$$

and

$$\begin{aligned}\|S_1^n x_n - T_1(PT_1)^{n-1}x_n\| &\leq \|S_1^n x_n - S_1^n y_n\| + \|S_1^n y_n - T_1(PT_1)^{n-1}y_n\| \\ &\quad + \|T_1(PT_1)^{n-1}y_n - T_1(PT_1)^{n-1}x_n\| \\ &= k_n^{(1)}\|x_n - y_n\| + \|S_1^n y_n + T_1(PT_1)^{n-1}y_n\| \\ &\quad + l_n^{(1)}\|y_n - x_n\|. \end{aligned} \tag{3.20}$$

Using (3.6), (3.19) and (3.20), we have

$$\lim_{n\to\infty} \|S_1^n x_n - T_1(PT_1)^{n-1}x_n\| = 0, \tag{3.21}$$

and so

$$\lim_{n\to\infty} \|x_n - T_1(PT_1)^{n-1}x_n\| = 0. \tag{3.22}$$

In addition,

$$\begin{aligned}\|x_{n+1} - S_1^n y_n\| &= \|P((1-\alpha_n)S_1^n y_n + \alpha_n T_1(PT_1)^{n-1}y_n) - P(S_1^n y_n)\| \\ &\leq (1-\alpha_n)\|S_1^n y_n - S_1^n y_n\| + \alpha_n\|T_1(PT_1)^{n-1}y_n - S_1^n y_n\|.\end{aligned}$$

Thus, it follows from (3.6) that

$$\lim_{n\to\infty} \|x_{n+1} - S_1^n y_n\| = 0. \tag{3.23}$$

In addition,

$$\|x_{n+1} - T_1(PT_1)^{n-1}y_n\| \leq \|x_{n+1} - S_1^n y_n\| + \|S_1^n y_n - T_1(PT_1)^{n-1}y_n\|.$$

By using (3.6) and (3.23), we have

$$\lim_{n\to\infty} \|x_{n+1} - T_1(PT_1)^{n-1}y_n\| = 0. \tag{3.24}$$

It follows from (3.21) and (3.22) that

$$\begin{aligned} \|S_1^n x_n - x_n\| &= \|S_1^n x_n - T_1(PT_1)^{n-1}x_n + T_1(PT_1)^{n-1}x_n - x_n\| \\ &\leq \|S_1^n x_n - T_1(PT_1)^{n-1}x_n\| + \|T_1(PT_1)^{n-1}x_n - x_n\| \\ &\to 0 \ (as\ n \to \infty). \end{aligned} \tag{3.25}$$

In addition,

$$\begin{aligned} \|S_1^n x_n - T_2(PT_2)^{n-1}x_n\| &= \|S_1^n x_n - x_n + x_n - T_2(PT_2)^{n-1}x_n\| \\ &\leq \|S_1^n x_n - x_n\| + \|x_n - T_2(PT_2)^{n-1}x_n\|. \end{aligned}$$

Thus, it follows from (3.16) and (3.25) that

$$\lim_{n\to\infty} \|S_1^n x_n - T_2(PT_2)^{n-1}x_n\| = 0. \tag{3.26}$$

In addition,

$$\begin{aligned} \|S_1^n y_n - T_2(PT_2)^{n-1}x_n\| &= \|S_1^n y_n - S_1^n x_n + S_1^n x_n - T_2(PT_2)^{n-1}x_n\| \\ &\leq \|S_1^n y_n - S_1^n x_n\| + \|S_1^n x_n - T_2(PT_2)^{n-1}x_n\| \\ &\leq k_n^{(1)}\|y_n - x_n\| + \|S_1^n x_n - T_2(PT_2)^{n-1}x_n\|. \end{aligned}$$

By using (3.19) and (3.26), we have

$$\lim_{n\to\infty} \|S_1^n y_n - T_2(PT_2)^{n-1}x_n\| = 0. \tag{3.27}$$

It follows from (3.23) and (3.27) that

$$\begin{aligned}\|x_{n+1}-T_2(PT_2)^{n-1}y_n\| &= \|x_{n+1}-S_1^n y_n+S_1^n y_n-T_2(PT_2)^{n-1}x_n\|\\ &= \|x_{n+1}-S_1^n y_n\|+\|S_1^n y_n-T_2(PT_2)^{n-1}x_n\|\\ &\to 0 \ (as\ n\to\infty). \end{aligned} \tag{3.28}$$

Again, since $(PT_i)(PT_i)^{n-2}y_{n-1}, x_n \in K$ for $i=1,2$ and T_1, T_2 are two asymptotically nonexpansive nonself-mappings, we have

$$\begin{aligned}\|T_i(PT_i)^{n-1}y_{n-1}-T_i x_n\| &= \|T_i((PT_i)(PT_i)^{n-2}y_{n-1})-T_i(Px_n)\|\\ &\le \max\{l_1^{(1)}, l_1^{(2)}\}\|(PT_i)(PT_i)^{n-2}y_{n-1}-Px_n\|\\ &\le \max\{l_1^{(1)}, l_1^{(2)}\}\|T_i(PT_i)^{n-2}y_{n-1}-x_n\|. \end{aligned} \tag{3.29}$$

Using (3.24) , (3.28) and (3.29), for $i=1,2$, we have

$$\lim_{n\to\infty}\|T_i(PT_i)^{n-1}y_{n-1}-T_i x_n\|=0. \tag{3.30}$$

Moreover, we have

$$\|x_{n+1}-y_n\|\le\|x_{n+1}-T_1(PT_1)^{n-1}y_n\|+\|T_1(PT_1)^{n-1}y_n-y_n\|.$$

Using (3.8) and (3.24), we have

$$\lim_{n\to\infty}\|x_{n+1}-y_n\|=0. \tag{3.31}$$

In addition, for $i=1,2$, we have

$$\begin{aligned}\|x_n-T_i x_n\| \le\ & \|x_n-T_i(PT_i)^{n-1}x_n\|+\|T_i(PT_i)^{n-1}x_n-T_i(PT_i)^{n-1}y_{n-1}\|\\ &+\|T_i(PT_i)^{n-1}y_{n-1}-T_i x_n\|\end{aligned}$$

$$\leq \|x_n - T_i(PT_i)^{n-1}x_n\| + max\{\sup_{n\geq 1} l_n^{(1)}, \sup_{n\geq 1} l_n^2\}\|x_n - y_{n-1}\|$$
$$+\|T_i(PT_i)^{n-1}y_{n-1} - T_ix_n\|.$$

Thus, it follows from (3.16), (3.22), (3.30) and (3.31) that

$$\lim_{n\to\infty} \|x_n - T_1x_n\| = \lim_{n\to\infty} \|x_n - T_2x_n\| = 0.$$

Finally, we prove that

$$\lim_{n\to\infty} \|x_n - S_1x_n\| = \lim_{n\to\infty} \|x_n - S_2x_n\| = 0.$$

In fact, for $i = 1, 2$, we have

$$\begin{aligned} \|x_n - S_ix_n\| &\leq \|x_n - T_i(PT_i)^{n-1}x_n\| + \|S_ix_n - T_i(PT_i)^{n-1}x_n\| \\ &\leq \|x_n - T_i(PT_i)^{n-1}x_n\| + \|S_1^nx_n - T_i(PT_i)^{n-1}x_n\|. \end{aligned}$$

Thus, it follows from (3.16), (3.21), (3.22) and (3.26) that

$$\lim_{n\to\infty} \|x_n - S_1x_n\| = \lim_{n\to\infty} \|x_n - S_2x_n\| = 0.$$

The proof is completed.

Now, we find two mapping, $S_1 = S_2 = S$ and $T_1 = T_2 = T$, satisfying the condition $\|x - T_iy\| \leq \|S_ix_n - T_iy\|$ for all $x, y \in K$ and $i = 1, 2$ in Lemma 3.2 as follows.

Example 3.1[13] Let $\mathbb{R}$ be the real line with the usual norm $|\cdot|$ and let $K = [-1, 1]$. Define two mappings $S, T : K \to K$ by

$$Tx = \begin{cases} -2\sin\frac{x}{2}, & \text{if } x \in [0, 1], \\ 2\sin\frac{x}{2}, & \text{if } x \in [-1, 0) \end{cases}$$

and

$$Sx = \begin{cases} x, & \text{if } x \in [0,1] \\ -x, & \text{if } x \in [-1,0). \end{cases}$$

Now, we show that T is nonexpansive. In fact, if $x, y \in [0,1]$ or $x, y \in [-1,0)$, than we have

$$|Tx - Ty| = 2|\sin\tfrac{x}{2} - \sin\tfrac{y}{2}| \leq |x - y|.$$

If $x \in [0,1]$ and $y \in [-1,0)$ or $x \in [-1,0)$ and $y \in [0,1]$, then we have

$$\begin{aligned} |Tx - Ty| &= 2|\sin\frac{x}{2} - \sin\frac{y}{2}| \\ &= 4|\sin\frac{x+y}{4}\cos\frac{x-y}{4}| \\ &\leq |x+y| \\ &\leq |x-y|. \end{aligned}$$

This implies that T is nonexpansive, and so T is an asymptotically nonexpansive mapping with $k_n = 1$ for each $n \geq 1$. Similarly, we can show that S is an asymptotically nonexpansive mapping with $l_n = 1$ for each $n \geq 1$.

Next, we consider the following cases:

Case 1. Let $x, y \in [0,1]$. Then we have

$$|x - Ty| = |x + 2\sin\tfrac{y}{2}| = |Sx - Ty|.$$

Case 2. Let $x, y \in [-1,0)$. Then we have

$$|x - Ty| = |x - 2\sin\tfrac{y}{2}| \leq |-x - 2\sin\tfrac{y}{2}| = |Sx - Ty|.$$

Case 3. Let $x \in [-1,0)$ and $y \in [0,1]$. Then we have

$$|x - Ty| = |x + 2\sin\tfrac{y}{2}| \leq |-x + 2\sin\tfrac{y}{2}| = |Sx - Ty|.$$

Case 4. Let $x \in [0,1]$ and $y \in [-1,0]$. Then we have

$$|x - Ty| = |x - 2\sin\tfrac{y}{2}| = |Sx - Ty|.$$

Theorem 3.1 Under the assumptions of Lemma 3.2, if one of S_1, S_2, T_1 and T_2 is completely continuous, then the sequence $\{x_n\}$ defined by (1.10) converges strongly to a common fixed point of S_1, S_2, T_1 and T_2.

Proof Without loss of generality, we can assume that S_1 is completely continuous. Since $\{x_n\}$ is bounded by Lemma 3.1, there exists a subsequence $\{S_1 x_{n_j}\}$ of $\{S_1 x_n\}$ such that $\{S_1 x_{n_j}\}$ converges strongly to some q^*. Moreover, we know that

$$\lim_{j\to\infty} \|x_{n_j} - S_1 x_{n_j}\| = \lim_{j\to\infty} \|x_{n_j} - S_2 x_{n_j}\| = 0$$

and

$$\lim_{j\to\infty} \|x_{n_j} - T_1 x_{n_j}\| = \lim_{j\to\infty} \|x_{n_j} - T_2 x_{n_j}\| = 0$$

by Lemma 3.2, which imply that

$$\|x_{n_j} - q^*\| \leq \|x_{n_j} - S_1 x_{n_j}\| + \|S_1 x_{n_j} - q^*\| \to 0$$

as $j \to \infty$, and so $x_{n_j} \to q^* \in K$. Thus, by the continuity of S_1, S_2, T_1 and T_2, we have

$$\|q^* - S_i q^*\| = \lim_{j\to\infty} \|x_{n_j} - S_i x_{n_j}\| = 0$$

and

$$\|q^* - T_i q^*\| = \lim_{j\to\infty} \|x_{n_j} - T_i x_{n_j}\| = 0$$

for $i = 1, 2$. Thus it follows that $q^* \in F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$. Furthermore, since $\lim_{n\to\infty} \|x_n - q^*\|$ exists by Lemma 3.1, we have $\lim_{n\to\infty} \|x_n - q^*\| = 0$. This completes the proof.

Theorem 3.2 Under the assumptions of Lemma 3.2, if one of S_1, S_2, T_1 and T_2 is semi-compact, then the sequence $\{x_n\}$ defined by (1.10) converges strongly to a common fixed point of S_1, S_2, T_1 and T_2.

Proof Since $\lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for $i = 1, 2$ by Lemma 3.2 and one of S_1, S_2, T_1 and T_2 is semi-compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some $q^* \in K$. Moreover, by the continuity of S_1, S_2, T_1 and T_2, we have $\|q^* - S_i q^*\| = \lim_{j\to\infty} \|x_{n_j} - S_i x_{n_j}\| = 0$ and $\|q^* - T_i q^*\| = \lim_{j\to\infty} \|x_{n_j} - T_i x_{n_j}\| = 0$ for $i = 1, 2$. Thus it follows that $q^* \in F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$. Since $\lim_{n\to\infty} \|x_n - q^*\|$ exists by Lemma 3.1, we have $\lim_{n\to\infty} \|x_n - q^*\| = 0$. This completes the proof.

Theorem 3.3 Under the assumptions of Lemma 3.2, if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$f(d(x, F)) \leq \|x - S_1 x\| + \|x - S_2 x\| + \|x - T_1 x\| + \|x - T_2 x\|$$

for all $x \in K$, where $F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$, then the sequence $\{x_n\}$ defined by (1.10) converges strongly to a common fixed point of S_1, S_2, T_1 and T_2.

Proof Since $\lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for $i = 1, 2$ by Lemma 3.2, we have $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ and $\lim_{n\to\infty} d(x_n, F)$ exists by Lemma 3.1, we have $\lim_{n\to\infty} d(x_n, F) = 0$.

Now, we show that $\{x_n\}$ is a Cauchy sequence in K. In fact, from (3.2), we have

$$\|x_{n+1} - q\| \leq (1 + (h_n^2 - 1))\|x_n - q\|$$

for each $n \geq 1$, where $h_n = \max\{k_n^{(1)},\ k_n^{(2)},\ l_n^{(1)},\ l_n^{(2)}\}$ and $q \in F$. For any

$m, n, m > n \geq 1$, we have

$$\begin{aligned} \|x_m - q\| &\leq (1 + (h_{m-1}^2 - 1))\|x_{m-1} - q\| \\ &\leq e^{h_{m-1-1}^2}\|x_{m-1} - q\| \\ &\leq e^{h_{m-1-1}^2} e^{h_{m-2-1}^2}\|x_{m-2} - q\| \\ &\vdots \\ &\leq e^{\sum_{i=n}^{m-1}(h_i^2 - 1)}\|x_n - q\| \\ &\leq M\|x_n - q\|, \end{aligned}$$

where $M = e^{\sum_{i=1}^{\infty}(h_i^2 - 1)}$. Thus, for any $q \in F$, we have

$$\begin{aligned} \|x_n - x_m\| &\leq \|x_n - q\| + \|x_m - q\| \\ &\leq (1 + M)\|x_n - q\|. \end{aligned}$$

Taking the infimum over all $q \in F$, we obtain

$$\|x_n - x_m\| \leq (1 + M)d(x_n, F).$$

Thus it follows from $\lim_{n\to\infty} d(x_n, F) = 0$ that $\{x_n\}$ is a Cauchy sequence. Since K is a closed subset of E, the sequence $\{x_n\}$ converges strongly to some $q^* \in K$. It is easy to prove that $F(S_1), F(S_2), F(T_1)$ and $F(T_2)$ are all closed and so F is a closed subset of K. Since $\lim_{n\to\infty} d(x_n, F) = 0$, $q^* \in F$, the sequence $\{x_n\}$ converges strongly to a common fixed point of S_1, S_2, T_1 and T_2. This completes the proof.

In the remainder of the section, we deal with the weak convergence of the iterative scheme (1.10) to a common fixed point of mixed type of two asymptotically nonexpansive self-mappings and two asymptotically nonexpansive nonself-mappings in uniformly convex Banach spaces.

Lemma 3.3 Under the assumptions of Lemma 3.1, for all $q_1, q_2 \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$, the limit

$$\lim_{n\to\infty} \|tx_n + (1-t)q_1 - q_2\|$$

exists for all $t \in [0,1]$, where $\{x_n\}$ is the sequence defined by (1.10).

Proof Set $a_n(t) = \lim_{n\to\infty} \|tx_n + (1-t)q_1 - q_2\|$. Then $\lim_{n\to\infty} a_n(0) = \|q_1 - q_2\|$ and, from Lemma 3.1, $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty} \|x_n - q_2\|$ exists. Thus it remains to prove Lemma 3.3 for any $t \in (0,1)$.

Define the mapping $G_n : K \to K \quad by$

$$\begin{aligned} G_n x &= P((1-\alpha_n)S_1^n P((1-\beta_n)S_2^n x + \beta_n T_2(PT_2)^{n-1}x) + \\ &\quad \alpha_n T_1(PT_1)^{n-1}P((1-\beta_n)S_2^n x + \beta_n T_2(PT_2)^{n-1}x)) \end{aligned}$$

for all $x \in K$. It follows that

$$\begin{aligned} \|G_n x - G_n y\| &= \|P((1-\alpha_n)S_1^n P((1-\beta_n)S_2^n x + \beta_n T_2(PT_2)^{n-1}x) + \\ &\quad \alpha_n T_1(PT_1)^{n-1}P((1-\beta_n)S_2^n x + \beta_n T_2(PT_2)^{n-1}x)) - \\ &\quad \|P((1-\alpha_n)S_1^n P((1-\beta_n)S_2^n y + \beta_n T_2(PT_2)^{n-1}y) + \\ &\quad \alpha_n T_1(PT_1)^{n-1}P((1-\beta_n)S_2^n y + \beta_n T_2(PT_2)^{n-1}y))\| \\ &\leq \|((1-\alpha_n)S_1^n P((1-\beta_n)S_2^n x + \beta_n T_2(PT_2)^{n-1}x) + \\ &\quad \alpha_n T_1(PT_1)^{n-1}P((1-\beta_n)S_2^n x + \beta_n T_2(PT_2)^{n-1}x)) - \\ &\quad \|((1-\alpha_n)S_1^n P((1-\beta_n)S_2^n y + \beta_n T_2(PT_2)^{n-1}y) + \\ &\quad \alpha_n T_1(PT_1)^{n-1}P((1-\beta_n)S_2^n y + \beta_n T_2(PT_2)^{n-1}y))\| \\ &= (1-\alpha_n)\|(S_1^n((1-\beta_n)S_2^n x + \beta_n T_2(PT_2)^{n-1}x) - \\ &\quad (S_1^n((1-\beta_n)S_2^n y + \beta_n T_2(PT_2)^{n-1}y) + \\ &\quad \alpha_n(T_1(PT_1)^{n-1}P((1-\beta_n)S_2^n x + \beta_n T_2(PT_2)^{n-1}x)) - \\ &\quad (T_1(PT_1)^{n-1}P((1-\beta_n)S_2^n y + \beta_n T_2(PT_2)^{n-1}y))\| \end{aligned}$$

$$\begin{aligned}
&\leq (1-\alpha_n)h_n\|((1-\beta_n)(S_2^n x - S_2^n y) + \beta_n T_2(PT_2)^{n-1}(x-y)\| + \\
&\quad \alpha_n h_n\|(1-\beta_n)(S_2^n x - S_2^n y) + \beta_n T_2(PT_2)^{n-1}(x-y)\| \\
&\leq (1-\alpha_n)h_n\|(1-\beta_n)(S_2^n x - S_2^n y)\| \\
&\quad +(1-\alpha_n)h_n\|\beta_n T_2(PT_2)^{n-1}(x-y)\| \\
&\quad +\alpha_n h_n\|(1-\beta_n)(S_2^n x - S_2^n y)\| + \alpha_n h_n\|\beta_n T_2(PT_2)^{n-1}(x-y)\| \\
&= (h_n^2 + h_n^2\beta_n - \alpha_n h_n^2 + h_n^2\alpha_n\beta_n)\|x-y\| + h_n^2\beta_n\|x-y\| \\
&\quad \alpha_n h_n\beta_n\|x-y\| + \alpha_n h_n^2(1-\beta_n)\|x-y\| + \alpha_n\beta_n h_n^2\|x-y\| \\
&= (h_n^2 + h_n^2\beta_n - \alpha_n h_n^2 + h_n^2\alpha_n\beta_n)\|x-y\| + h_n^2\beta_n\|x-y\| \\
&\quad \alpha_n h_n^2\beta_n\|x-y\| + +\alpha_n\beta_n h_n^2\|x-y\| \\
&= h_n^2\|x-y\| \qquad (3.32)
\end{aligned}$$

for all $x, y \in K$, where $h_n = max\{k_n^{(1)}, k_n^{(2)}, l_n^{(1)}, l_n^{(2)}\}$. Letting $h_n = 1 + v_n$, it follows from $1 \leq \prod_{j=n}^{\infty} h_j^2 \leq e^{2\sum_{j=n}^{\infty} v_j}$ and $\sum_{n=1}^{\infty} v_n < \infty$ that $\lim_{n\to\infty} \prod_{j=n}^{\infty} h_j^2 = 1$. Setting

$$S_{n,m} = G_{n+m-1}G_{n+m-2...}G_n \qquad (3.33)$$

for each $m \geq 1$, from (3.32) and (3.33), it follows that

$$\|S_{n,m}x - S_{n,m}y\| \ (\prod_{j=n}^{n+m-1} h_j^2)\|x-y\|$$

for all $x, y \in K$ and $S_{n,m}x_n = x_{n+m}, S_{n,m}q = q$ for any $q \in F$. Let

$$b_{n,m} = \|tS_{n,m}x_n + (1-t)S_{n,m}q_1 - S_{m,n}(tx_n + (1-t)q_1)\|. \qquad (3.34)$$

Then, using (3.34) and Lemma 2.4, we have

$$\begin{aligned}
b_{n,m} &\leq (\prod_{j=n}^{n+m-1} h_j^2)\gamma^{-1}(\|x_n - q_1\| - (\prod_{j=n}^{n+m-1} h_j^2)^{-1}\|S_{n,m}x_n - S_{n,m}q_1\|) \\
&\leq (\prod_{j=n}^{\infty} h_j^2)\gamma^{-1}(\|x_n - q_1\| - (\prod_{j=n}^{\infty} h_j^2)^{-1}\|x_{n,m} - q_1\|).
\end{aligned}$$

It follows from Lemma 3.1 and $\lim_{n\to\infty}\prod_{j=n}^{\infty} h_j^2 = 1$ that $\lim_{n\to\infty} b_{n,m} = 0$ uniformly for all m. Observe that

$$\begin{aligned} a_{n,m}(t) &\le \|S_{n,m}(tx_n + (1-t)q_1) - q_2\| + b_{n,m} \\ &= \|S_{n,m}(tx_n + (1-t)q_1) - S_{n,m}q_2\| + b_{n,m} \\ &\le (\prod_{j=n}^{n+m-1} h_j^2)\|tx_n + (1-t)q_1 - q_2\| + b_{n,m} \\ &\le (\prod_{j=n}^{\infty} h_j^2) a_n(t) + b_{n,m}. \end{aligned}$$

Thus we have $\limsup_{n\to\infty} a_n(t) \le \liminf_{n\to\infty} a_n(t)$, That is, $\lim_{n\to\infty}\|tx_n + (1-t)q_1 - q_2\|$ exists for all $t \in (0,1)$. This completes the proof.

Lemma 3.4 *Under the assumptions of Lemma 3.1, if E has a $Fre\acute{c}het$ differentiable norm, then, for all $q_1, q_2 \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$, the limit*

$$\lim_{n\to\infty}\langle x_n, j(q_1 - q_2)\rangle$$

exists, where $\{x_n\}$ is the sequence defined by (1.10). Furthermore, if $Ww(\{x_n\})$ denotes the set of all weak subsequential limits of $\{x_n\}$, then $\langle x^* - y^*, j(q_1 - q_2)\rangle = 0$ for all $q_1, q_2 \in F$ and $x^*, y^* \in Ww(\{x_n\})$.

Proof This follows basically as in the proof of Lemma 3.2 of [8] using Lemma 3.3 instead of Lemma 3.1 of [8].

Theorem 3.4 Under the assumptions of Lemma 3.2, if E has Fréchet differentiable norm, then the sequence $\{x_n\}$ defined by (1.10) converges weakly to a common fixed point of S_1, S_2, T_1 and T_2.

Proof Since E is a uniformly convex Banach space the sequence $\{x_n\}$ is bounded by Lemma 3.1, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some $q \in K$. By Lemma 3.2, we have

$$\lim_{k\to\infty} \|x_{n_k} - S_i x_{n_k}\| = \lim_{k\to\infty} \|x_{n_k} - T_i x_{n_k}\| = 0$$

for $i = 1.2$. It follows Lemma 2.3 that $q \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$. Now, we prove that the sequence $\{x_n\}$ converges weakly to q. Suppose that there exists a subsequence $\{x_{m_j}\}$ of $\{x_n\}$ such that $\{x_{m_j}\}$ converges weakly to some $q_1 \in K$. Then, by the same method given we can also prove that $q_1 \in F$. So, $q_1, q_2 \in F \ \cap Ww(\{x_n\})$. It follows from Lemma 3.4 that

$$\|q - q_1\|^2 = \langle q - q_1, j(q - q_1)\rangle = 0.$$

Therefore, $q_1 = q$, which shows that the sequence $\{x_n\}$ converges weakly to q. This completes the proof.

Theorem 3.5 Under the assumptions of Lemma 3.2, if the dual space $E^\star$ of E has the Kadce-Klee property, then sequence $\{x_n\}$ defined by (1.10) converges weakly to a common fixed point of S_1, S_2, T_1 and T_2.

Proof Using the same method given in Theorem 3.4, we can prove that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some $q \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(S_2)$. Now, we prove that the sequence $\{x_n\}$ converges weakly to q. Suppose that there exists a subsequence $\{x_{m_j}\}$ of $\{x_n\}$ such that $\{x_{m_j}\}$ converges weakly to some $q^* \in K$. Then, as for q, we have $q^* \in F$. It follows from Lemma 3.3 that the limit

$$\lim_{n\to\infty} \|tx_n - (1-t)q - q^*\|$$

exists for all $t \in [0,1]$. Again, since $q, q^* \in Ww(\{x_n\}), q^* = q$ be Lemma 2.5. This shows that the sequence $\{x_n\}$ converges weakly to q. This completes the proof.

Theorem 3.6 Under the assumptions of Lemma 3.2, if E satisfies Opial's condition, then the sequence $\{x_n\}$ defined by (1.10) converges weakly to a common fixed point of S_1, S_2, T_1 and T_2.

Proof Using the same method as given in Theorem 3.4, we can prove that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some $q \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(S_2)$. Now, we prove that the sequence $\{x_n\}$ converges weakly to q. Suppose that there exists a subsequence $\{x_{m_j}\}$ of $\{x_n\}$ such that $\{x_{m_j}\}$ converges weakly to some $\overline{q} \in K$ and $\overline{q} \neq q$. Then, as for q, we have $\overline{q} \in F$. Using Lemma 3.1, we have the following two limits exist:

$$\lim_{n\to\infty} \|x_n - q\| = c, \qquad \lim_{n\to\infty} \|x_n - \overline{q}\| = c_1.$$

Thus, by Opial's condition, we have

$$\begin{aligned} c &= \limsup_{k\to\infty} \|x_{n_k} - q\| \\ &< \limsup_{k\to\infty} \|x_{n_k} - \overline{q}\| \\ &= \limsup_{j\to\infty} \|x_{m_j} - \overline{q}\| \\ &< \limsup_{j\to\infty} \|x_{m_j} - q\| = c, \end{aligned}$$

which is contradiction, and so $q = \overline{q}$. This shows that the sequence $\{x_n\}$ converges weakly to q. This completes the proof.

CHAPTER 4
Conclusions

In this chapter, we will present the main results obtained in this research.

4.1 Conclusions

Lemma 4.1 Let E be a real uniformly convex Banach space and K a nonempty closed convex nonexpansive retract of E with P as a nonexpansive retraction. Let S_1, S_2 : $K \longrightarrow K$ be two asymptotically nonexpansive self-mappings with$\{k_n^{(1)}\}$, $\{k_n^{(2)}\} \subset [1, \infty)$ and T_1, T_2 : $K \to E$ be two asymptotically nonexpansive nonself-mappings with $\{l_n^{(1)}\}$, $\{l_n^{(2)}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)} - 1) < \infty$ for $i = 1$, 2, respectively and $F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1)$. From an arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ using (1.10) Then

(1) $\lim\limits_{n\to\infty} \|x_n - q\|$ exists for any $q \in F$;

(2) $\lim\limits_{n\to\infty} d(x_n, F)$ exists.

Lemma 4.2 Let E ba a real uniformly convex Banach space and K a nonempty closed convex nonexpansive retract of E with P as a nonexpansive retraction. Let $S_1, S_2 : K \to K$ be two asymptotically nonexpansive self-mappings with $\{k_n^{(1)}\}, \{k_n^{(2)}\} \subset [1, \infty)$ and $T_1, T_2 : K \to E$ be two asymptotically nonexpansive nonself-mappings with $\{l_n^{(1)}\}, \{l_n^{(2)}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)} - 1) < \infty$ for $i = 1, 2$, respectively and $F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$. From an arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ using (1.10). If $\|x - T_i y\| \leq \|S_i x - T_i y\|$ for all $x, y \in K$ and $i = 1, 2$, then $\lim\limits_{n\to\infty} \|x_n - S_i x_n\| = \lim\limits_{n\to\infty} \|x_n - T_i x_n\| = 0$ for $i = 1, 2$.

Theorem 4.1 Under the assumptions of Lemma 4.2, if one of S_1, S_2, T_1 and T_2 is completely continuous, then the sequence $\{x_n\}$ defined by (1.10) converges strongly to a common fixed point of S_1, S_2, T_1 and T_2.

Theorem 4.2 Under the assumptions of Lemma 4.2, if one of S_1, S_2, T_1 and T_2 is semi-compact, then the sequence $\{x_n\}$ defined by (1.10) converges strongly to a common fixed point of S_1, S_2, T_1 and T_2.

Theorem 4.3 Under the assumptions of Lemma 4.2, if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$f(d(x, F)) \leq \|x - S_1 x\| + \|x - S_2 x\| + \|x - T_1 x\| + \|x - T_2 x\|$$

for all $x \in K$, where $F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$, then the sequence $\{x_n\}$ defined by (1.10) converges strongly to a common fixed point of S_1, S_2, T_1 and T_2.

Lemma 4.3 Under the assumptions of Lemma 4.1, for all $q_1, q_2 \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$, the limit

$$\lim_{n\to\infty} \|tx_n + (1-t)q_1 - q_2\|$$

exists for all $t \in [0, 1]$, where $\{x_n\}$ is the sequence defined by (1.10).

Theorem 4.4 Under the assumptions of Lemma 4.2, if E has Fréchet differentiable norm, then the sequence $\{x_n\}$ defined by (1.10) converges weakly to a common fixed point of S_1, S_2, T_1 and T_2.

Theorem 4.5 Under the assumptions of Lemma 4.2, if the dual space $E^\star$ of E has the Kadce-Klee property, then sequence $\{x_n\}$ defined by (1.10) converges weakly to a common fixed point of S_1, S_2, T_1 and T_2.

Theorem 4.6 Under the assumptions of Lemma 4.2, if E satisfies Opial's condition, then the sequence $\{x_n\}$ defined by (1.10) converges weakly to a common fixed point of S_1, S_2, T_1 and T_2.

BIBLIOGRAPHY

[1] Chang S.S., Cho Y.J., Zhou H.: Demi-closed principle and weak convergence problems for asymptotically nonexpansive mappings, J. Korean Math. Soc. 38 (2001) 1245-1260.

[2] Chidume C.E., Ofoedu E.U., Zegeye H.: Strong and weak convergence theorems for asymptotically nonexpansive mappings. J. Math. Anal.Appl. 280 (2003) 364-374.

[3] Falset J.G., Kaczor W., Kuczumow T., Reich S.: Weak convergence theorems for asymptotically nonexpansive mappings and semigroups. Nonlinaear Anal. 43 (2001) 377-401.

[4] Goebel K., Kirk W.A.: A fixed point theorem for asymptotically nonexpansive mapping. Proc. Am. Math. Soc 35 (1972) 171-174.

[5] Goebel K., Kirk W.A.: Topics in Metric Fixed Point Theory. Cambridge Studies in Advanced Mathematics. vol. 28. Cambridge University Press, Cambridge (1990).

[6] Guo W., Cho Y.J.: On strong convergence of the implicit iterative processes with error for a finite family of asymptotically nonexpansive mappings. Appl. Math. Lett. 21 (2008) 1046-1052.

[7] Guo W., Gho Y.J, Guo W.: Convergence theorems for mixed type asymptotically nonexpansive mappings, Fixed Point Theory Appl. 2012, DOI : 10. 1186 / 1687-1812-2012-2211(2012).

[8] Guo W., Guo W.: Weak convergence theorems for asymptotically nonexpansive nonself-mappings. Appl. Math. Lett. 24 (2011) 2181-2185.

[9] Ishikawa S.: Fixed points and iteration of nonexpansive mappings of in a Banach spaces, Proc. Amer. Math. Soc. 73 (1976) 61-71.

[10] Jung J.S., Kim S.S.: Strong convergence theorem for nonexpansive nonself-mapping in Banach space, Nonlinear Anal. 33 (1998) 321-329.

[11] Khan S.H., Fukhar-ud-din H.: Weak and strong convergence of a scheme with errors for two nonexpansive mappings, Nonlinear Anal. 61 (2005) 1295-1301.

[12] Liu Q.: Iterative sequences for asymptotically quasi-nonexpansive mappings. J. Math. Anal. Appl. 259 (2001) 1-7.

[13] Liu Z., Feng C., Ume J.S., Kang S.M.: Weak and strong convergence for common fixed points of a pair of nonexpansive and asymptotically nonexpansive mappings. Taiwan. J. Math. 11 (2007) 27-42.

[14] Matsushita S.Y., Kuroiwa D.: Strong convergence of averaging iteration of nonexpansive nonself-mappings, J. Math. Anal. Appl. 294 (2004) 206-214.

[15] Opial Z.: Weak convergence of successive approximations for nonexpansive mappings. Bull. Am. Math. Soc. 73 (1967) 591-597.

[16] Osilike M.O., Aniagbosor S.C.: Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings, Math. Comput. Modelling 32 (2000) 1181-1191.

[17] Osilike M.O., Udomene A.: Demiclosedness principle and convergence theorems for strictly pseudocontractive mappings of Browder-Petryshyn type. J. Math. Anal. Appl. 256 (2001) 431-445.

[18] Pathak H.K., Cho Y.J., Kang S.M.: Strong and weak convergence theorems for nonself-asymptotically perturbed nonexpansive mappings. Nonlinear Anal. 70 (2009) 1929-1938.

[19] Razani A., Salahifard H.: Invariant approximation for CAT(0) spaces. Nonlinear Anal. 72 (2010) 2421-2425.

[20] Rhoades B.E.: Fixed point itertions for certain nonlinear mappings, J. Math. Anal. Appl. 183 (1994) 118-120.

[21] Reich S.: Weak convergence theorems for onexpansive mappings in Banach spaces, J. Math. Anal. Appl. 183 (1994) 118-120.

[22] Schu J.: Iterative construction of a fixed points of asymptotically nonexpansive mappings, J. Math. Anal. Appl. 67 (1979) 274-276.

[23] Schu J.: Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull. Math. Soc. 43 (1991) 153-159.

[24] Semple C.: Phylogenetics. Oxford Lecture Series in Mathematics and Its Application. Oxford University Press, Oxford (2003).

[25] Shahzad N.: Approximating fixed points of non-self nonexpansive mappings in Banach spaces, Nonlinear Anal.61 (2005) 1031-1039.

[26] Sun Z.H.: Strong convergence of an implicit iteration process for a finite family of asymptrotically quasi-nonexpansive mappings. J. Math. Anal. Appl. 286 (2003) 351-358.

[27] Takahashi W., Kim G.E.: Strong convergence of approximants to fixed points of nonexpansive nonself-mappings, Nonlinear Anal.32 (1998) 447-454.

[28] Tan K.K., Xu H.K.: Approximating fixed points of nonexpansive mapping by the Ishikawa iteration process, J. Math. Anal. Appl. 178 (1993) 301-308.

[29] Tan K.K., Xu H.K.: Fixed point iteration process for asymptotically nonexpansive mappings. Proc. Amer. Math. Xoc. 122(3) (1994) 733-739.

[30] Thianwan S.: Common fixe point of new iteration for two asymptotically nonnexpansive nonself-mapping in a Banach space. Feb. 15 (2009) 688-695.

[31] Wang L.: Strong and weak convergence theorems for common fixed points of nonself asymptotically nonexpansive mappings, J. Math. Anal. Appl. 323 (2006) 550-557.

[32] Xu H.K., Yin X.M.: Strong convergence theorems for nonexpansive nonself-mappings, Nonlinear Anal. 242 (1995) 23-228.

[33] Zhao-hong Sun: Strong convergence of an implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings. J. Math. Anal. Appl. 286 (2003) 351-358.

APPENDIX

Convergence of projection type iterative processes of mixed asymptotically nonexpansive mappings

Ruethaita yeampun, Wiliphorn witoon, Intira kapang, Tanakit thianwan *

School of Science, University of Phayao, Phayao 56000, Thailand

Abstract

In this research, we introduce a projection type iterative scheme of mixed type for two asymptotically nonexpansive self-mappings and two asymptotically nonexpansive nonself-mappings in uniformly convex Banach spaces. Weak and Strong convergence theorems are established in uniformly convex Banach spaces.

Keywords: mixed asymptotically nonexpansive mapping; strong and weak convergence; common fixed point; uniformly convex Banach space.

AMS Subject Classification: 47Ho4, 47H10, 54H25.

1 Introduction

Let K be a nonempty closed convex subset of a real normed linear space E. A self-mapping $T : K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. A self-mapping $T : K \to K$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{1.1}$$

for all $x, y \in K$ and $n \geq 1$.

A mapping $T : K \to K$ is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L \|x - y\| \tag{1.2}$$

*Corresponding author.

Email addresses: ruethaita2537@gmail.com (R.yeampun), popza55@hotmail.com (W.witoon), kratai.in2304@hotmail.com (I. kapang,), tanakit.th@up.ac.th (T. Thianwan)

for all $x, y \in K$ and $n \geq 1$.

It is easy to see that if T is an asymptotically nonexpansive, then it is uniformly L-Lipschitzian with the uniform Lipschitz constant $L = sup\{k_n : n \geq 1\}$.

Fixed-point iteration process for nonexpansive self-mappings including Mann and Ishikawa iteration processes have been studied extensively by various authors [1, 9, 11, 16, 17, 22].
For nonexpansive nonself-mappings, some authors [10, 14, 25, 27, 32] have studied the strong and weak convergence theorems in Hilbert space or uniformly convex Banach space. In 1972, Goebel and Kirk [4] introduced the class of asymptotically nonexpansive self-mappings, who proved that if K is nonempty closed convex subset of real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping on C, then T has a fixed point.

In 1991, Schu [23] introduced a modified Mann iteration process to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbret space. More precisely, he proved the following theorem.

Theorem 1.1 (see [23]). Let H be a Hilbert space, and let K be a nonempty closed convex and bounded subset of H. Let $T : K \to K$ be an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$ for all $n \geq 1$, $lim_{n\to\infty} k_n = 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}$ be a sequence in $[0.1]$ satisfying the condition $0 < a \leq \alpha_n \leq b < 1, n \geq 1$, for some constant a, b. Then the sequence $\{x_n\}$ generated from an arbitrary $x_1 \in K$ by the relation

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \qquad n \geq 1, \tag{1.3}$$

converges strongly to some fixed point of T.

Since then, Schu's iteration process has been widely used to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert or Banach
spaces (see [16],[20],[21],[23],[28]).

The concept of asymptotically nonexpansive nonself-mappings was introduced by Chidume, Ofoedu, and Zegeye [4] in 2003 as the generlization of asymptotically nonexpansive self-mappings. The nonself of asymptotically nonexpansive nonself-mapping is defined as follows.

Definition 1.2 (see [4]). Let K be a nonempty subset of a real normed linear space E. Let $P : E \to K$ be a nonexpansive retraction of E onto K. A nonself-mapping $T : K \to E$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ as $n \to \infty$ such that

$$\|T(PT)^{n-1}x - T(PT)_1^{n-1}y\| \leq k_n\|x - y\| \tag{1.4}$$

for all $x, y \in K$ and $n \geq 1$. A non-self-mapping T is said to be uniformly $L - Lipschitzian$ if there exists a constant $L \geq 0$ such that

$$\|T(PT)^{n-1}x - T(PT)_1^{n-1}y\| \leq L\|x - y\| \tag{1.5}$$

for all $x, y \in K$ and $n \geq 1$.

We denote by $(PT)^0$ the identity map from K onto itself. In [2], the authors studied the following iterative sequence: $x_1 \in K$,

$$x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \tag{1.6}$$

to approximate some fixed point of T under suitable conditions.

If T is a self-mapping, then P becomes the identity mapping so that (1.4) and (1.5) reduce to (1.1) and (1.2), respectively, and (1.6) reduces to (1.3).

In 2006, Wang[31] generalized the iteration process (1.8) as follows: $x_1 \in K$,

$$\begin{aligned} y_n &= P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n), \\ x_{n+1} &= P((1-\alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n),\ n \geq 1, \end{aligned} \tag{1.7}$$

where $T_1, T_2 : K \to E$ are asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in [0,1). He proved strong and weak convergence of the sequence $\{\alpha_n\}$ defined by (1.7) to a common fixed point of T_1 and T_2 under appropriate conditions. Meanwhile, the results of [31] generalized the results of [2].

In 2009, a new iterative scheme which is called the projection type Ishikawa iteration for two asymptotically nonexpansive nonself-mappings was defined and constructed by Thianwan [30]. It is given as follows:

$$\begin{aligned} y_n &= P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n), \\ x_{n+1} &= P((1-\alpha_n)y_n + \alpha_n T_1(PT_1)^{n-1}y_n),\ n \geq 1, \end{aligned} \tag{1.8}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate real sequences in [0,1). He studied the scheme for two asymptotically nonexpansive nonself-mappings and proved strong and weak convergence of the sequences $\{x_n\}$ and $\{y_n\}$ to a common fixed point of T_1, T_2 under suitable conditions in a uniformly convex Banach space.

Note that Thianwan process (1.8) and Wang process (1.7) are independent neither reduces to the other.

If $T_1 = T_2$ and $\beta_n = 0$ for all $n \geq 1$, then (1.8) reduces to (1.6). It also can be reduces to Schu process (1.3).

Recently, Guo, Cho and Guo [7] studied the following iteration scheme: $x_1 \in K$,

$$\begin{aligned} y_n &= P((1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1}x_n), \\ x_{n+1} &= P((1-\alpha_n)S_1^n x_n + \alpha_n T_1(PT_1)^{n-1}y_n),\ n \geq 1, \end{aligned} \tag{1.9}$$

where $S_1, S_2 : K \to K$ are asymptotically nonexpansive self-mappings, $T_1, T_2 : K \to E$ are asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in [0,1). They studied the strong and weak convergence of the iterative scheme (1.9) under proper conditions.

If S_1 and S_2 are the identity mappings, then the iterative scheme (1.9) reduces to the scheme (1.7).

Motivated by these recent works, we introduce and study a new iterative scheme in this paper. The scheme is defind as follows.

Let E be a real Banach space, K be a nonempty closed convex subset of E and $P : E \to K$ be a nonexpansive retraction of E onto K. Let $S_1, S_2 : K \to K$ be two asymptotically nonexpansive self-mappings and $T_1, T_2 : K \to E$ be two asymptotically nonexpansive nonself-mappings. Then, we define the new iteration scheme of mixed type as follows : $x_1 \in K$,

$$\begin{aligned} y_n &= P((1-\beta_n)S_2^n x_n + \beta_n T_2(PT)^{n-1}x_n), \\ x_{n+1} &= P((1-\alpha_n)S_1^n y_n + \alpha_n T_1(PT)^{n-1}y_n),\ n \geq 1, \end{aligned} \tag{1.10}$$

where $\{\alpha_n\}, \{\beta_n\}$ are two sequences [0,1).

The iterative scheme (1.10) is called the projective type iterative process for mixed type of asymptotically nonexpansive mappings. If S_1 and S_2 are the identity mappings, then the iterative scheme (1.10) reduces to (1.8).

Note that (1.9) and (1.10) are independent neither reduces to the other.

The purpose of this paper is to construct an iteration scheme for approximating common fixed points of two asymptotically nonexpansive self-mappings and two asymptotically nonexpansive nonself-mappings and to prove some strong and weak convergence theorems for such mappings in a real uniformly convex Banach space.

2 Preliminaries

We denote the set of common fixed points of S_1, S_2, T_1 and T_2 by $F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$ and denote the distance between a point z and a set A in E by $d(z, A) = \inf_{x \in A} \|z - x\|$.

Now, we recall some well-known concepts and results.

Let E be a real Banach space, E^* be the dual space of E and $J : E \to 2^{E^*}$ be the *normalized duality mapping* defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|\|f\|, \|f\| = \|x\|\}$$

for all $x \in E$, where $\langle \cdot, \cdot \rangle$ denotes duality pairing between E and E^*. A single-valued normalized duality mapping is denoted by j.

A subset K of a real Banach space E is called a *retract* of E [2] if there exists a continuous mapping $P : E \to K$ such that $Px = x$ for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : E \to E$ is called a *retraction* if $P^2 = P$. It follows that if a mapping P is a retraction, than $Py = y$ for all y in the range of P.

Recall that a Banach space E is said to satisfy *Opial′s condition* [15] if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ implying that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

A mapping $T : K \to E$ is said to be semi-compact if, for any sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $x^* \in K$.

A Banach space E is said to have a *Fre´chet differentiable* norm [17] if, for all $x \in \mathcal{U} = \{x \in E : \|x\| = 1\}$,

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists and is attained uniformly in $y \in \mathcal{U}$.

A Banach space E is said to have the *Kadec-Klee property* [5] if for every sequence $\{x_n\}$ in $E, x_n \to x$ weakly and $\|x_n\| \to \|x\|$, if follows that $x_n \to x$ strongly.

In the sequel, the following lemmas are needed to prove our main results.

Lemma 2.1 [26] *Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be three nonnegative sequences satisfying the following condition*:

$$a_{n+1} \leq (1 + b_n)a_n + c_n$$

for each $n \geq n_0$, where n_0 is some nonnegative integer, $\sum_{n=n_0}^{\infty} b_n < \infty$ and $\sum_{n=n_0}^{\infty} c_n < \infty$. Then $lim_{n\to\infty} a_n$ exists.

Lemma 2.2 [23] *Let E be a real uniformly convex Banach space and $0 < p \leq t_n < q < 1$ for each $n \geq 1$. Also, suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of such that*

$$\limsup_{n\to\infty} \|x_n\| \leq r, \quad \limsup_{n\to\infty} \|y_n\| \leq r, \quad \lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = r$$

hold for some $r \geq 0$. Then $lim_{n\to\infty}\|x_n - y_n\| = 0$.

Lemma 2.3 [2] *Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E and $T : K \to E$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$. Then $I - T$ is demiclosed at zero, i.e., if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in F(T)$, where $F(T)$ is the set of fixed points of T.*

Lemma 2.4 [3] *Let E be a uniformly convex Banach space and K be a convex subset of E. Then there exists a strictly increasing continuous convex function $\gamma : [0, \infty) \to [0, \infty)$ with $\gamma(0) = 0$ such* *that, for each mapping $S : K \to K$ with a Lipschitz constant $L > 0$,*

$$\|\alpha_n + (1-\alpha)Sy - S(\alpha x + (1-\alpha)y)\| \le L\gamma^1(\|x-y\| - \tfrac{1}{L}\|Sx - Sy\|)$$

for all $x, y \in K$ and $0 < \alpha < 1$.

Lemma 2.5 [3] *Let E be a uniformly convex Banach space such that its dual space E^* has the Kadec-Klee property. Suppose $\{x_n\}$ is a bounded sequence and $f_1, f_2 \in W_w(\{x_n\})$ such that*

$$\lim_{n\to\infty} \|\alpha x_n + (1-\alpha)f_1 - f_2\|$$

exists for all $\alpha \in [0, 1]$, where $W_w(\{x_n\})$ denotes the set of all weak subsequential limits of $\{x_n\}$. Then $f_1 = f_2$.

3 Main results

In this chapter, we prove theorems of strong and weak convergence of the iterative scheme given in (1.10) to a common fixed point of mixed type of two asymptotically nonexpansive self-mappings and two asymptotically nonexpansive nonself-mappings in uniformly convex Banach spaces.

In order to prove our main results, the following lemmas are needed.

Lemma 3.1 Let E be a real uniformly convex Banach space and K a nonempty closed convex nonexpansive retract of E with P as a nonexpansive retraction. Let $S_1, S_2 : K \longrightarrow K$ be two asymptotically nonexpansive self-mappings with$\{k_n^{(1)}\}$, $\{k_n^{(2)}\} \subset [1, \infty)$ and $T_1, T_2 : K \to E$ be two asymptotically nonexpansive nonself-mappings with $\{l_n^{(1)}\}$, $\{l_n^{(2)}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)} - 1) < \infty$ for $i = 1, 2$, respectively and $F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1)$. From an arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ using (1.10) Then

(1) $\lim_{n\to\infty} \|x_n - q\|$ exists for any $q \in F$;
(2) $\lim_{n\to\infty} d(x_n, F)$ exists.

Proof Let $q \in F$. Setting $h_n = max\{k_n^{(1)},\ k_n^{(2)},\ l_n^{(1)},\ l_n^{(2)}\}$. Using (1.10), we have

$$\begin{aligned}
\|y_n - q\| &= \|P((1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1}x_n) - q\| \\
&= \|P((1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1}x_n) - P(q)\| \\
&\le \|(1-\beta_n)(S_2^n x_n - q) + \beta_n(T_2(PT_2)^{n-1}x_n - q)\| \\
&\le (1-\beta_n)h_n\|x_n - q\| + \beta_n h_n\|x_n - q\| \\
&= h_n\|x_n - q\|, \qquad (3.1)
\end{aligned}$$

and so

$$
\begin{aligned}
\|x_{n+1}-q\| &= \|P((1-\alpha_n)S_1^n y_n + \alpha_n T_1(PT_1)^{n-1}y_n) - q\| \\
&= \|P((1-\alpha_n)S_1^n y_n + \alpha_n T_1(PT_1)^{n-1}y_n) - P(q)\| \\
&\leq \|(1-\alpha_n)(S_1^n y_n - q) + \alpha_n(T_1(PT_1)^{n-1}y_n - q)\| \\
&\leq (1-\alpha_n)h_n\|y_n - q\| + \alpha_n h_n\|y_n - q\| \\
&= h_n\|y_n - q\| \\
&\leq h_n^2\|x_n - q\| \\
&= (1+(h_n^2-1))\|x_n - q\|. \qquad (3.2)
\end{aligned}
$$

Since $\sum_{n=1}^{\infty}(k_n^{(i)}-1) < \infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)}-1) < \infty$ for $i = 1,2$, we have $\sum_{n=1}^{\infty}(h_n^2-1) < \infty$. It follows from Lemma 2.1 that $\lim_{n\to\infty}\|x_n - q\|$ exists.

(2) Taking the infimum over all $q \in F$ in (3.2), we have

$$d(x_{n+1}, F) \leq (1+(h_n^2-1))d(x_n, F)$$

for each $n \geq 1$. It follows from $\sum_{n=1}^{\infty}(h_n^2-1) < \infty$ and Lemma 2.1 that the conclusion (2) holds. This completes the proof.

Lemma 3.2 Let E be a real uniformly convex Banach space and K a nonempty closed convex nonexpansive retract of E with P as a nonexpansive retraction. Let $S_1, S_2 : K \to K$ be two asymptotically nonexpansive self-mappings with $\{k_n^{(1)}\}, \{k_n^{(2)}\} \subset [1,\infty)$ and $T_1, T_2 : K \to E$ be two asymptotically nonexpansive nonself-mappings with $\{l_n^{(1)}\}, \{l_n^{(2)}\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)}-1) < \infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)}-1) < \infty$ for $i = 1,2$, respectively and $F = F(S_1)\cap F(S_2)\cap F(T_1)\cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. From an arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ using (1.10). If $\|x - T_i y\| \leq \|S_i x - T_i y\|$ for all $x, y \in K$ and $i = 1,2$, then $\lim_{n\to\infty}\|x_n - S_i x_n\| = \lim_{n\to\infty}\|x_n - T_i x_n\| = 0$ for $i = 1,2$.

Proof Suppose that $\|x - T_i y\| \leq \|S_i x - T_i y\|$ for all $x, y \in K$ and $i = 1,2$. Let $q \in F$. Set $h_n = max\{k_n^{(1)}, k_n^{(2)}, l_n^{(1)}, l_n^{(2)}\}$. By Lemma 3.1, we are that $lim_{n\to\infty}\|x_n - q\|$ exists. Assume that $\lim_{n\to\infty}\|x_n - q\| = c$. Since $\sum_{n=1}^{\infty}(h_n^2-1) < \infty$ and $\lim_{n\to\infty}\|x_{n+1} - q\| = c$, letting $n \to \infty$ in the inequality (3.2), we have

$$\lim_{n\to\infty}\|(1-\alpha_n)(S_1^n y_n - q) + \alpha_n T_1(PT_1)^{n-1}y_n - q)\| = c. \qquad (3.3)$$

In addition, $\|S_1^n y_n - q\| \leq k_n^{(1)}\|y_n - q\|$, taking the lim sup on both sides in this inequality, we have

$$\limsup_{n\to\infty}\|S_1^n y_n - q\| \leq c. \qquad (3.4)$$

Taking the lim sup on both sides in the inequality (3.1), we obtain $\limsup_{n\to\infty} \|y_n - q\| \le c$, and so

$$\limsup_{n\to\infty} \|T_1(PT_1)^{n-1}y_n - q\| \le \limsup_{n\to\infty} l_n^{(1)}\|y_n - q\| \le c. \tag{3.5}$$

By using (3.3), (3.4), (3.5) and Lemma 2.2, we have

$$\lim_{n\to\infty} \|S_1^n y_n - T_1(PT_1)^{n-1}y_n\| = 0. \tag{3.6}$$

Since

$$\|y_n - T_1(PT_1)^{n-1}y_n\| \le \|S_1^n y_n - T_1(PT_1)^{n-1}y_n\|. \tag{3.7}$$

Letting $n \to \infty$ in the inequality (3.7), by (3.6), we have

$$\lim_{n\to\infty} \|y_n - T_1(PT_1)^{n-1}y_n\| = 0. \tag{3.8}$$

From (3.2), we have

$$\|x_{n+1} - q\| \quad \le \quad h_n\|y_n - q\| \le h_n^2\|y_n - q\|. \tag{3.9}$$

Taking the lim inf on both sidies in the inequality (3.9), we have

$$\liminf_{n\to\infty} \|y_n - q\| \ge c. \tag{3.10}$$

Since $\limsup_{n\to\infty} \|y_n - q\| \le c$, by (3.10), we have $\lim_{n\to\infty} \|y_n - q\| = c$. This implies that

$$\begin{aligned} c = \lim \|y_n - q\| &\le \lim_{n\to\infty} \|(1-\beta_n)(S_2^n x_n - q) + \beta_n(T_2(PT_2)^{n-1}x_n - q)\| \\ &\le \lim_{n\to\infty} \|x_n - q\| = c, \end{aligned}$$

and so

$$\lim_{n\to\infty} \|(1-\beta_n)(S_2^n x_n - q) + \beta_n(T_2(PT_2)^{n-1}x_n - q)\| = c. \tag{3.11}$$

In addition, we have

$$\limsup_{n\to\infty} \|S_2^n x_n - q\| \le \limsup_{n\to\infty} k_n^{(2)}\|x_n - q\| = c \tag{3.12}$$

and

$$\limsup_{n\to\infty} \|T_2(PT_2)^{n-1}x_n - q\| \le \limsup_{n\to\infty} l_n^{(2)}\|x_n - q\| = c. \tag{3.13}$$

It follows from (3.11), (3.12), (3.13) and Lemma 2.2 that

$$\lim_{n\to\infty} \|S_2^n x_n - T_2(PT_2)^{n-1}x_n\| = 0. \tag{3.14}$$

Now, we prove that

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = \lim_{n\to\infty} \|x_n - T_2 x_n\| = 0.$$

Indeed, since $\|x_n - T_2(PT_2)^{n-1}x_n\| \le \|S_2^n x_n - T_2(PT_2)^{n-1}x_n\|$. (3.15)

Using (3.14) and (3.15), we have

$$\lim_{n\to\infty} \|x_n - T_2(PT_2)^{n-1}x_n\| = 0. \tag{3.16}$$

Since $S_2^n x_n = P(S_2^n x_n)$ and $P : E \to K$ is nonexpansive rectraction of E onto K, we have

$$\begin{aligned} \|y_n - S_2^n x_n\| &\le \|(1-\beta_n)(S_2^n x_n - S_2^n x_n) + \beta_n(T_2(PT_2)^{n-1}x_n - S_2^n x_n)\| \\ &\le \beta_n \|T_2(PT_2)^{n-1}x_n - S_2^n x_n\|. \end{aligned}$$

Using (3.14), we have

$$\lim_{n\to\infty} \|y_n - S_2^n x_n\| = 0. \tag{3.17}$$

Furthermore, we have

$$\begin{aligned} \|y_n - x_n\| \le \|y_n - S_2^n x_n\| &+ \|S_2^n x_n - T_2(PT_2)^{n-1}x_n\| \\ &+ \|T_2(PT_2)^{n-1}x_n - x_n\|. \end{aligned} \tag{3.18}$$

It follows from (3.14), (3.16), (3.17) and (3.18) that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.19}$$

Since

$$\|x_n - T_1(PT_1)^{n-1}x_n\| \le \|S_1^n x_n - T_1(PT_1)^{n-1}x_n\|$$

and

$$\begin{aligned} \|S_1^n x_n - T_1(PT_1)^{n-1}x_n\| &\le \|S_1^n x_n - S_1^n y_n\| + \|S_1^n y_n - T_1(PT_1)^{n-1}y_n\| \\ &\quad + \|T_1(PT_1)^{n-1}y_n - T_1(PT_1)^{n-1}x_n\| \\ &= k_n^{(1)}\|x_n - y_n\| + \|S_1^n y_n + T_1(PT_1)^{n-1}y_n\| \\ &\quad + l_n^{(1)}\|y_n - x_n\|. \end{aligned} \tag{3.20}$$

Using (3.6), (3.19) and (3.20), we have

$$\lim_{n\to\infty} \|S_1^n x_n - T_1(PT_1)^{n-1}x_n\| = 0, \tag{3.21}$$

and so

$$\lim_{n\to\infty} \|x_n - T_1(PT_1)^{n-1}x_n\| = 0. \tag{3.22}$$

In addition,

$$\begin{aligned} \|x_{n+1} - S_1^n y_n\| &= \|P((1-\alpha_n)S_1^n y_n + \alpha_n T_1(PT_1)^{n-1}y_n) - P(S_1^n y_n)\| \\ &\leq (1-\alpha_n)\|S_1^n y_n - S_1^n y_n\| + \alpha_n\|T_1(PT_1)^{n-1}y_n - S_1^n y_n\|. \end{aligned}$$

Thus, it follows from (3.6) that

$$\lim_{n\to\infty} \|x_{n+1} - S_1^n y_n\| = 0. \tag{3.23}$$

In addition,

$$\|x_{n+1} - T_1(PT_1)^{n-1}y_n\| \leq \|x_{n+1} - S_1^n y_n\| + \|S_1^n y_n - T_1(PT_1)^{n-1}y_n\|.$$

By using (3.6) and (3.23), we have

$$\lim_{n\to\infty} \|x_{n+1} - T_1(PT_1)^{n-1}y_n\| = 0. \tag{3.24}$$

It follows from (3.21) and (3.22) that

$$\begin{aligned} \|S_1^n x_n - x_n\| &= \|S_1^n x_n - T_1(PT_1)^{n-1}x_n + T_1(PT_1)^{n-1}x_n - x_n\| \\ &\leq \|S_1^n x_n - T_1(PT_1)^{n-1}x_n\| + \|T_1(PT_1)^{n-1}x_n - x_n\| \\ &\to 0 \ (as\ n \to \infty). \end{aligned} \tag{3.25}$$

In addition,

$$\begin{aligned} \|S_1^n x_n - T_2(PT_2)^{n-1}x_n\| &= \|S_1^n x_n - x_n + x_n - T_2(PT_2)^{n-1}x_n\| \\ &\leq \|S_1^n x_n - x_n\| + \|x_n - T_2(PT_2)^{n-1}x_n\|. \end{aligned}$$

Thus, it follows from (3.16) and (3.25) that

$$\lim_{n\to\infty} \|S_1^n x_n - T_2(PT_2)^{n-1}x_n\| = 0. \tag{3.26}$$

In addition,

$$\begin{aligned}\|S_1^n y_n - T_2(PT_2)^{n-1}x_n\| &= \|S_1^n y_n - S_1^n x_n + S_1^n x_n - T_2(PT_2)^{n-1}x_n\| \\ &\le \|S_1^n y_n - S_1^n x_n\| + \|S_1^n x_n - T_2(PT_2)^{n-1}x_n\| \\ &\le k_n^{(1)}\|y_n - x_n\| + \|S_1^n x_n - T_2(PT_2)^{n-1}x_n\|.\end{aligned}$$

By using (3.19) and (3.26), we have

$$\lim_{n\to\infty} \|S_1^n y_n - T_2(PT_2)^{n-1}x_n\| = 0. \tag{3.27}$$

It follows from (3.23) and (3.27) that

$$\begin{aligned}\|x_{n+1} - T_2(PT_2)^{n-1}y_n\| &= \|x_{n+1} - S_1^n y_n + S_1^n y_n - T_2(PT_2)^{n-1}x_n\| \\ &= \|x_{n+1} - S_1^n y_n\| + \|S_1^n y_n - T_2(PT_2)^{n-1}x_n\| \\ &\to 0 \ (as\ n \to \infty).\end{aligned} \tag{3.28}$$

Again, since $(PT_i)(PT_i)^{n-2}y_{n-1}, x_n \in K$ for $i = 1, 2$ and T_1, T_2 are two asymptotically nonexpansive nonself-mappings, we have

$$\begin{aligned}\|T_i(PT_i)^{n-1}y_{n-1} - T_i x_n\| &= \|T_i((PT_i)(PT_i)^{n-2}y_{n-1}) - T_i(Px_n)\| \\ &\le \max\{l_1^{(1)}, l_1^{(2)}\}\|(PT_i)(PT_i)^{n-2}y_{n-1} - Px_n\| \\ &\le \max\{l_1^{(1)}, l_1^{(2)}\}\|T_i(PT_i)^{n-2}y_{n-1} - x_n\|.\end{aligned} \tag{3.29}$$

Using (3.24) , (3.28) and (3.29), for $i = 1, 2$, we have

$$\lim_{n\to\infty} \|T_i(PT_i)^{n-1}y_{n-1} - T_i x_n\| = 0. \tag{3.30}$$

Moreover, we have

$$\|x_{n+1} - y_n\| \le \|x_{n+1} - T_1(PT_1)^{n-1}y_n\| + \|T_1(PT_1)^{n-1}y_n - y_n\|.$$

Using (3.8) and (3.24), we have

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{3.31}$$

In addition, for $i = 1, 2$, we have

$$\begin{aligned}\|x_n - T_i x_n\| \le{}& \|x_n - T_i(PT_i)^{n-1}x_n\| + \|T_i(PT_i)^{n-1}x_n - T_i(PT_i)^{n-1}y_{n-1}\| \\ &+\|T_i(PT_i)^{n-1}y_{n-1} - T_i x_n\| \\ \le{}& \|x_n - T_i(PT_i)^{n-1}x_n\| + max\{\sup_{n\ge1} l_n^{(1)}, \sup_{n\ge1} l_n^{2}\}\|x_n - y_{n-1}\| \\ &+\|T_i(PT_i)^{n-1}y_{n-1} - T_i x_n\|.\end{aligned}$$

Thus, it follows from (3.16), (3.22), (3.30) and (3.31) that

$$\lim_{n\to\infty} \|x_n - T_1x_n\| = \lim_{n\to\infty} \|x_n - T_2x_n\| = 0.$$

Finally, we prove that

$$\lim_{n\to\infty} \|x_n - S_1x_n\| = \lim_{n\to\infty} \|x_n - S_2x_n\| = 0.$$

In fact, for $i = 1, 2$, we have

$$\begin{aligned} \|x_n - S_ix_n\| &\le \|x_n - T_i(PT_i)^{n-1}x_n\| + \|S_ix_n - T_i(PT_i)^{n-1}x_n\| \\ &\le \|x_n - T_i(PT_i)^{n-1}x_n\| + \|S_1^nx_n - T_i(PT_i)^{n-1}x_n\|. \end{aligned}$$

Thus, it follows from (3.16), (3.21), (3.22) and (3.26) that

$$\lim_{n\to\infty} \|x_n - S_1x_n\| = \lim_{n\to\infty} \|x_n - S_2x_n\| = 0.$$

The proof is completed.

Now, we find two mapping, $S_1 = S_2 = S$ and $T_1 = T_2 = T$, satisfying the condition $\|x - T_iy\| \le \|S_ix_n - T_iy\|$ for all $x, y \in K$ and $i = 1, 2$ in Lemma 3.2 as follows.

Example 3.1[13] Let $\mathbb{R}$ be the real line with the usual norm $|\cdot|$ and let $K = [-1, 1]$. Define two mappings $S, T : K \to K$ by

$$Tx = \begin{cases} -2\sin\frac{x}{2}, & \text{if } x \in [0, 1], \\ 2\sin\frac{x}{2}, & \text{if } x \in [-1, 0) \end{cases}$$

and

$$Sx = \begin{cases} x, & \text{if } x \in [0, 1] \\ -x, & \text{if } x \in [-1, 0). \end{cases}$$

Now, we show that T is nonexpansive. In fact, if $x, y \in [0, 1]$ or $x, y \in [-1, 0)$, than we have

$$|Tx - Ty| = 2|\sin\tfrac{x}{2} - \sin\tfrac{y}{2}| \le |x - y|.$$

If $x \in [0, 1]$ and $y \in [-1, 0)$ or $x \in [-1, 0)$ and $y \in [0, 1]$, then we have

$$\begin{aligned} |Tx - Ty| &= 2|\sin\frac{x}{2} - \sin\frac{y}{2}| \\ &= 4|\sin\frac{x+y}{4}\cos\frac{x-y}{4}| \\ &\le |x + y| \\ &\le |x - y|. \end{aligned}$$

This implies that T is nonexpansive, and so T is an asymptotically nonexpansive mapping with $k_n = 1$ for each $n \geq 1$. Similarly, we can show that S is an asymptotically nonexpansive mapping with $l_n = 1$ for each $n \geq 1$.

Next, we consider the following cases:

Case 1. Let $x, y \in [0, 1]$. Then we have

$|x - Ty| = |x + 2\sin\frac{y}{2}| = |Sx - Ty|$.

Case 2. Let $x, y \in [-1, 0)$. Then we have

$|x - Ty| = |x - 2\sin\frac{y}{2}| \leq |-x - 2\sin\frac{y}{2}| = |Sx - Ty|$.

Case 3. Let $x \in [-1, 0)$ and $y \in [0, 1]$. Then we have

$|x - Ty| = |x + 2\sin\frac{y}{2}| \leq |-x + 2\sin\frac{y}{2}| = |Sx - Ty|$.

Case 4. Let $x \in [0, 1]$ and $y \in [-1, 0]$. Then we have

$|x - Ty| = |x - 2\sin\frac{y}{2}| = |Sx - Ty|$.

Theorem 3.1 Under the assumptions of Lemma 3.2, if one of S_1, S_2, T_1 and T_2 is completely continuous, then the sequence $\{x_n\}$ defined by (1.10) converges strongly to a common fixed point of S_1, S_2, T_1 and T_2.

Proof Without loss of generality, we can assume that S_1 is completely continuous. Since $\{x_n\}$ is bounded by Lemma 3.1, there exists a subsequence $\{S_1 x_{n_j}\}$ of $\{S_1 x_n\}$ such that $\{S_1 x_{n_j}\}$ converges strongly to some q^*. Moreover, we know that

$$\lim_{j\to\infty} \|x_{n_j} - S_1 x_{n_j}\| = \lim_{j\to\infty} \|x_{n_j} - S_2 x_{n_j}\| = 0$$

and

$$\lim_{j\to\infty} \|x_{n_j} - T_1 x_{n_j}\| = \lim_{j\to\infty} \|x_{n_j} - T_2 x_{n_j}\| = 0$$

by Lemma 3.2, which imply that

$$\|x_{n_j} - q^*\| \leq \|x_{n_j} - S_1 x_{n_j}\| + \|S_1 x_{n_j} - q^*\| \to 0$$

as $j \to \infty$, and so $x_{n_j} \to q^* \in K$. Thus, by the continuity of S_1, S_2, T_1 and T_2, we have

$$\|q^* - S_i q^*\| = \lim_{j\to\infty} \|x_{n_j} - S_i x_{n_j}\| = 0$$

and

$$\|q^* - T_i q^*\| = \lim_{j\to\infty} \|x_{n_j} - T_i x_{n_j}\| = 0$$

for $i = 1, 2$. Thus it follows that $q^* \in F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$. Furthermore, since $\lim_{n\to\infty} \|x_n - q^*\|$ exists by Lemma 3.1, we have $\lim_{n\to\infty} \|x_n - q^*\| = 0$. This completes the proof.

Theorem 3.2 Under the assumptions of Lemma 3.2, if one of S_1, S_2, T_1 and T_2 is semi-compact, then the sequence $\{x_n\}$ defined by (1.10) converges strongly to a common fixed point of S_1, S_2, T_1 and T_2.

Proof Since $\lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for $i = 1, 2$ by Lemma 3.2 and one of S_1, S_2, T_1 and T_2 is semi-compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some $q^* \in K$. Moreover, by the continuity of S_1, S_2, T_1 and T_2, we have $\|q^* - S_i q^*\| = \lim_{j\to\infty} \|x_{n_j} - S_i x_{n_j}\| = 0$ and $\|q^* - T_i q^*\| = \lim_{j\to\infty} \|x_{n_j} - T_i x_{n_j}\| = 0$ for $i = 1, 2$. Thus it follows that $q^* \in F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$. Since $\lim_{n\to\infty} \|x_n - q^*\|$ exists by Lemma 3.1, we have $\lim_{n\to\infty} \|x_n - q^*\| = 0$. This completes the proof.

Theorem 3.3 Under the assumptions of Lemma 3.2, if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$f(d(x, F)) \le \|x - S_1 x\| + \|x - S_2 x\| + \|x - T_1 x\| + \|x - T_2 x\|$$

for all $x \in K$, where $F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$, then the sequence $\{x_n\}$ defined by (1.10) converges strongly to a common fixed point of S_1, S_2, T_1 and T_2.

Proof Since $\lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for $i = 1, 2$ by Lemma 3.2, we have $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ and $\lim_{n\to\infty} d(x_n, F)$ exists by Lemma 3.1, we have $\lim_{n\to\infty} d(x_n, F) = 0$.

Now, we show that $\{x_n\}$ is a Cauchy sequence in K. In fact, from (3.2), we have

$$\|x_{n+1} - q\| \le (1 + (h_n^2 - 1))\|x_n - q\|$$

for each $n \ge 1$, where $h_n = \max\{k_n^{(1)},\ k_n^{(2)},\ l_n^{(1)},\ l_n^{(2)}\}$ and $q \in F$. For any $m, n, m > n \ge 1$, we have

$$\begin{aligned}
\|x_m - q\| &\le (1 + (h_{m-1}^2 - 1))\|x_{m-1} - q\| \\
&\le e^{h_{m-1}^2 - 1}\|x_{m-1} - q\| \\
&\le e^{h_{m-1}^2 - 1} e^{h_{m-2}^2 - 1}\|x_{m-2} - q\| \\
&\vdots \\
&\le e^{\sum_{i=n}^{m-1}(h_i^2 - 1)}\|x_n - q\| \\
&\le M\|x_n - q\|,
\end{aligned}$$

where $M = e^{\sum_{i=1}^{\infty}(h_i^2 - 1)}$. Thus, for any $q \in F$, we have

$$\begin{aligned}
\|x_n - x_m\| &\le \|x_n - q\| + \|x_m - q\| \\
&\le (1 + M)\|x_n - q\|.
\end{aligned}$$

Taking the infimum over all $q \in F$, we obtain

$$\|x_n - x_m\| \le (1 + M)d(x_n, F).$$

Thus it follows from $\lim_{n\to\infty} d(x_n, F) = 0$ that $\{x_n\}$ is a Cauchy sequence. Since K is a closed subset of E, the sequence $\{x_n\}$ converges strongly to some $q^* \in K$. It is easy to prove that $F(S_1), F(S_2), F(T_1)$ and $F(T_2)$ are all closed and so F is a closed subset of K. Since $\lim_{n\to\infty} d(x_n, F) = 0$, $q^* \in F$, the sequence $\{x_n\}$ converges strongly to a common fixed point of S_1, S_2, T_1 and T_2. This completes the proof.

The remainder of the section, we deal with the weak convergence of the iterative scheme (1.10) to a common fixed point of mixed type of two asymptotically nonexpansive self-mappings and two asymptotically nonexpansive nonself-mappings in uniformly convex Banach spaces.

Lemma 3.3 Under the assumptions of Lemma 3.1, for all $q_1, q_2 \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$, the limit

$$\lim_{n\to\infty} \|tx_n + (1-t)q_1 - q_2\|$$

exists for all $t \in [0,1]$, where $\{x_n\}$ is the sequence defined by (1.10).

Proof Set $a_n(t) = \lim_{n\to\infty} \|tx_n + (1-t)q_1 - q_2\|$. Then $\lim_{n\to\infty} a_n(0) = \|q_1 - q_2\|$ and, from Lemma 3.1, $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty} \|x_n - q_2\|$ exists. Thus it remains to prove Lemma 3.3 for any $t \in (0,1)$. Define the mapping $G_n : K \to K$ *by*

$$\begin{aligned} G_n x &= P((1-\alpha_n)S_1^n P((1-\beta_n)S_2^n x + \beta_n T_2(PT_2)^{n-1}x) + \\ &\quad \alpha_n T_1(PT_1)^{n-1} P((1-\beta_n)S_2^n x + \beta_n T_2(PT_2)^{n-1}x)) \end{aligned}$$

for all $x \in K$. It follows that

$$\begin{aligned} \|G_n x - G_n y\| &= \|P((1-\alpha_n)S_1^n P((1-\beta_n)S_2^n x + \beta_n T_2(PT_2)^{n-1}x) + \\ &\quad \alpha_n T_1(PT_1)^{n-1} P((1-\beta_n)S_2^n x + \beta_n T_2(PT_2)^{n-1}x)) - \\ &\quad \|P((1-\alpha_n)S_1^n P((1-\beta_n)S_2^n y + \beta_n T_2(PT_2)^{n-1}y) + \\ &\quad \alpha_n T_1(PT_1)^{n-1} P((1-\beta_n)S_2^n y + \beta_n T_2(PT_2)^{n-1}y))\| \\ &\le \|((1-\alpha_n)S_1^n P((1-\beta_n)S_2^n x + \beta_n T_2(PT_2)^{n-1}x) + \\ &\quad \alpha_n T_1(PT_1)^{n-1} P((1-\beta_n)S_2^n x + \beta_n T_2(PT_2)^{n-1}x)) - \\ &\quad \|((1-\alpha_n)S_1^n P((1-\beta_n)S_2^n y + \beta_n T_2(PT_2)^{n-1}y) + \\ &\quad \alpha_n T_1(PT_1)^{n-1} P((1-\beta_n)S_2^n y + \beta_n T_2(PT_2)^{n-1}y))\| \\ &= (1-\alpha_n)\|(S_1^n((1-\beta_n)S_2^n x + \beta_n T_2(PT_2)^{n-1}x) - \\ &\quad (S_1^n((1-\beta_n)S_2^n y + \beta_n T_2(PT_2)^{n-1}y) + \\ &\quad \alpha_n(T_1(PT_1)^{n-1} P((1-\beta_n)S_2^n x + \beta_n T_2(PT_2)^{n-1}x)) - \\ &\quad (T_1(PT_1)^{n-1} P((1-\beta_n)S_2^n y + \beta_n T_2(PT_2)^{n-1}y))\| \end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n)h_n\|((1-\beta_n)(S_2^n x - S_2^n y) + \beta_n T_2(PT_2)^{n-1}(x-y)\| + \\
&\quad \alpha_n h_n\|(1-\beta_n)(S_2^n x - S_2^n y) + \beta_n T_2(PT_2)^{n-1}(x-y)\| \\
&\leq (1-\alpha_n)h_n\|(1-\beta_n)(S_2^n x - S_2^n y)\| \\
&\quad +(1-\alpha_n)h_n\|\beta_n T_2(PT_2)^{n-1}(x-y)\| \\
&\quad +\alpha_n h_n\|(1-\beta_n)(S_2^n x - S_2^n y)\| + \alpha_n h_n\|\beta_n T_2(PT_2)^{n-1}(x-y)\| \\
&= (h_n^2 + h_n^2\beta_n - \alpha_n h_n^2 + h_n^2\alpha_n\beta_n)\|x-y\| + h_n^2\beta_n\|x-y\| \\
&\quad \alpha_n h_n\beta_n\|x-y\| + \alpha_n h_n^2(1-\beta_n)\|x-y\| + \alpha_n\beta_n h_n^2\|x-y\| \\
&= (h_n^2 + h_n^2\beta_n - \alpha_n h_n^2 + h_n^2\alpha_n\beta_n)\|x-y\| + h_n^2\beta_n\|x-y\| \\
&\quad \alpha_n h_n^2\beta_n\|x-y\| + +\alpha_n\beta_n h_n^2\|x-y\| \\
&= h_n^2\|x-y\| \qquad (3.32)
\end{aligned}
$$

for all $x, y \in K$, where $h_n = max\{k_n^{(1)}, k_n^{(2)}, l_n^{(1)}, l_n^{(2)}\}$. Letting $h_n = 1 + v_n$, it follows from $1 \leq \prod_{j=n}^{\infty} h_j^2 \leq e^{2\sum_{j=n}^{\infty} v_j}$ and $\sum_{n=1}^{\infty} v_n < \infty$ that $\lim_{n\to\infty} \prod_{j=n}^{\infty} h_j^2 = 1$. Setting

$$S_{n,m} = G_{n+m-1}G_{n+m-2...}G_n \qquad (3.33)$$

for each $m \geq 1$, from (3.32) and (3.33), it follows that

$$\|S_{n,m}x - S_{n,m}y\| \; (\prod_{j=n}^{n+m-1} h_j^2)\|x-y\|$$

for all $x, y \in K$ and $S_{n,m}x_n = x_{n+m}, S_{n,m}q = q$ for any $q \in F$. Let

$$b_{n,m} = \|tS_{n,m}x_n + (1-t)S_{n,m}q_1 - S_{m,n}(tx_n + (1-t)q_1)\|. \qquad (3.34)$$

Then, using (3.34) and Lemma 2.4, we have

$$
\begin{aligned}
b_{n,m} &\leq (\prod_{j=n}^{n+m-1} h_j^2)\gamma^{-1}(\|x_n - q_1\| - (\prod_{j=n}^{n+m-1} h_j^2)^{-1}\|S_{n,m}x_n - S_{n,m}q_1\|) \\
&\leq (\prod_{j=n}^{\infty} h_j^2)\gamma^{-1}(\|x_n - q_1\| - (\prod_{j=n}^{\infty} h_j^2)^{-1}\|x_{n,m} - q_1\|).
\end{aligned}
$$

It follows from Lemma 3.1 and $\lim_{n\to\infty} \prod_{j=n}^{\infty} h_j^2 = 1$ that $\lim_{n\to\infty} b_{n,m} = 0$

uniformly for all m. Observe that

$$\begin{aligned} a_{n,m}(t) &\leq \|S_{n,m}(tx_n + (1-t)q_1) - q_2\| + b_{n,m} \\ &= \|S_{n,m}(tx_n + (1-t)q_1) - S_{n,m}q_2\| + b_{n,m} \\ &\leq (\prod_{j=n}^{n+m-1} h_j^2)\|tx_n + (1-t)q_1 - q_2\| + b_{n,m} \\ &\leq (\prod_{j=n}^{\infty} h_j^2)a_n(t) + b_{n,m}. \end{aligned}$$

Thus we have $\limsup_{n\to\infty} a_n(t) \leq \liminf_{n\to\infty} a_n(t)$, That is, $\lim_{n\to\infty} \|tx_n+(1-t)q_1-q_2\|$ exists for all $t \in (0,1)$. This completes the proof.

Lemma 3.4 *Under the assumptions of Lemma 3.1, if E has a Fre̒chet differentiable norm, then, for all $q_1, q_2 \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$, the limit*

$$\lim_{n\to\infty} \langle x_n, j(q_1 - q_2)\rangle$$

exists, where $\{x_n\}$ is the sequence defined by (1.10). Furthermore, if $Ww(\{x_n\})$ denotes the set of all weak subsequential limits of $\{x_n\}$, then $\langle x^* - y^*, j(q_1 - q_2)\rangle = 0$ for all $q_1, q_2 \in F$ and $x^*, y^* \in Ww(\{x_n\})$.

Proof This follows basically as in the proof of Lemma 3.2 of [12] using Lemma 3.3 instead of Lemma 3.1 of [8].

Theorem 3.4 Under the assumptions of Lemma 3.2, if E has Fréchet differentiable norm, then the sequence $\{x_n\}$ defined by (1.10) converges weakly to a common fixed point of S_1, S_2, T_1 and T_2.

Proof Since E is a uniformly convex Banach space the sequence $\{x_n\}$ is bounded by Lemma 3.1, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some $q \in K$. By Lemma 3.2, we have

$$\lim_{k\to\infty} \|x_{n_k} - S_i x_{n_k}\| = \lim_{k\to\infty} \|x_{n_k} - T_i x_{n_k}\| = 0$$

for $i = 1.2$. It follows Lemma 2.3 that $q \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$.
Now, we prove that the sequence $\{x_n\}$ converges weakly to q. Suppose that there exists a subsequence $\{x_{m_j}\}$ of $\{x_n\}$ such that $\{x_{m_j}\}$ converges weakly to some $q_1 \in K$. Then, by the same method given above can also prove that $q_1 \in F$. So, $q_1, q_2 \in F \cap Ww(\{x_n\})$. It follows from Lemma 3.4 that

$$\|q - q_1\|^2 = \langle q - q_1, j(q - q_1)\rangle = 0.$$

Therefore, $q_1 = q$, which shows that the sequence $\{x_n\}$ converges weakly to q. This completes the proof.

Theorem 3.5 Under the assumptions of Lemma 3.2, if the dual space $E^\star$ of E has the Kadce-Klee property, then sequence $\{x_n\}$ defined by (1.10) converges weakly to a common fixed point of S_1, S_2, T_1 and T_2.

Proof Using the same method given in Theorem 3.4, we can prove that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some $q \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(S_2)$. Now, we prove that the sequence $\{x_n\}$ converges weakly to q. Suppose that there exists a subsequence $\{x_{m_j}\}$ of $\{x_n\}$ such that $\{x_{m_j}\}$ converges weakly to some $q^* \in K$. Then, as for q, we have $q^* \in F$. It follows from Lemma 3.3 that the limit

$$\lim_{n\to\infty} \|tx_n - (1-t)q - q^*\|$$

exists for all $t \in [0,1]$. Again, since $q, q^* \in Ww(\{x_n\}), q^* = q$ be Lemma 2.5. This shows that the sequence $\{x_n\}$ converges weakly to q. This completes the proof.

Theorem 3.6 Under the assumptions of Lemma 3.2, if E satisfies Opial's condition, then the sequence $\{x_n\}$ defined by (1.10) converges weakly to a common fixed point of S_1, S_2, T_1 and T_2.

Proof Using the same method as given in Theorem 3.4, we can prove that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some $q \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(S_2)$. Now, we prove that the sequence $\{x_n\}$ converges weakly to q. Suppose that there exists a subsequence $\{x_{m_j}\}$ of $\{x_n\}$ such that $\{x_{m_j}\}$ converges weakly to some $\overline{q} \in K$ and $\overline{q} \neq q$. Then, as for q, we have $\overline{q} \in F$. Using Lemma 3.1, we have the following two limits exist:

$$\lim_{n\to\infty} \|x_n - q\| = c, \qquad \lim_{n\to\infty} \|x_n - \overline{q}\| = c_1.$$

Thus, by Opial's condition, we have

$$\begin{aligned} c &= \limsup_{k\to\infty} \|x_{n_k} - q\| \\ &< \limsup_{k\to\infty} \|x_{n_k} - \overline{q}\| \\ &= \limsup_{j\to\infty} \|x_{m_j} - \overline{q}\| \\ &< \limsup_{j\to\infty} \|x_{m_j} - q\| = c, \end{aligned}$$

which is contradiction, and so $q = \overline{q}$. This shows that the sequence $\{x_n\}$ converges weakly to q. This completes the proof.

References

[1] Chang S.S., Cho Y.J., Zhou H.: Demi-closed principle and weak convergence problems for asymptotically nonexpansive mappings, J. Korean Math. Soc. 38 (2001) 1245-1260.

[2] Chidume C.E., Ofoedu E.U., Zegeye H.: Strong and weak convergence theorems for asymptotically nonexpansive mappings. J. Math. Anal.Appl. 280 (2003) 364-374.

[3] Falset J.G., Kaczor W., Kuczumow T., Reich S.: Weak convergence theorems for asymptotically nonexpansive mappings and semigroups. Nonlinaear Anal. 43 (2001) 377-401.

[4] Goebel K., Kirk W.A.: A fixed point theorem for asymptotically nonexpansive mapping. Proc. Am. Math. Soc 35 (1972) 171-174.

[5] Goebel K., Kirk W.A.: Topics in Metric Fixed Point Theory. Cambridge Studies in Advanced Mathematics. vol. 28. Cambridge University Press, Cambridge (1990).

[6] Guo W., Cho Y.J.: On strong convergence of the implicit iterative processes with error for a finite family of asymptotically nonexpansive mappings. Appl. Math. Lett. 21 (2008) 1046-1052.

[7] Guo W., Gho Y.J, Guo W.: Convergence theorems for mixed type asymptotically nonexpansive mappings, Fixed Point Theory Appl. 2012, DOI : 10. 1186 / 1687-1812-2012-2211(2012).

[8] Guo W., Guo W.: Weak convergence theorems for asymptotically nonexpansive nonself-mappings. Appl. Math. Lett. 24 (2011) 2181-2185.

[9] Ishikawa S.: Fixed points and iteration of nonexpansive mappings of in a Banach spaces, Proc. Amer. Math. Soc. 73 (1976) 61-71.

[10] Jung J.S., Kim S.S.: Strong convergence theorem for nonexpansive nonself-mapping in Banach space, Nonlinear Anal. 33 (1998) 321-329.

[11] Khan S.H., Fukhar-ud-din H.: Weak and strong convergence of a scheme with errors for two nonexpansive mappings, Nonlinear Anal. 61 (2005) 1295-1301.

[12] Liu Q.: Iterative sequences for asymptotically quasi-nonexpansive mappings. J. Math. Anal. Appl. 259 (2001) 1-7.

[13] Liu Z., Feng C., Ume J.S., Kang S.M.: Weak and strong convergence for common fixed points of a pair of nonexpansive and asymptotically nonexpansive mappings. Taiwan. J. Math. 11 (2007) 27-42.

[14] Matsushita S.Y., Kuroiwa D.: Strong convergence of averaging iteration of nonexpansive nonself-mappings, J. Math. Anal. Appl. 294 (2004) 206-214.

[15] Opial Z.: Weak convergence of successive approximations for nonexpansive mappings. Bull. Am. Math. Soc. 73 (1967) 591-597.

[16] Osilike M.O., Aniagbosor S.C.: Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings, Math. Comput. Modelling 32 (2000) 1181-1191.

[17] Osilike M.O., Udomene A.: Demiclosedness principle and convergence theorems for strictly pseudocontractive mappings of Browder-Petryshyn type. J. Math. Anal. Appl. 256 (2001) 431-445.

[18] Pathak H.K., Cho Y.J., Kang S.M.: Strong and weak convergence theorems for nonself-asymptotically perturbed nonexpansive mappings. Nonlinear Anal. 70 (2009) 1929-1938.

[19] Razani A., Salahifard H.: Invariant approximation for CAT(0) spaces. Nonlinear Anal. 72 (2010) 2421-2425.

[20] Rhoades B.E.: Fixed point itertions for certain nonlinear mappings, J. Math. Anal. Appl. 183 (1994) 118-120.

[21] Reich S.: Weak convergence theorems for onexpansive mappings in Banach spaces, J. Math. Anal. Appl. 183 (1994) 118-120.

[22] Schu J.: Iterative construction of a fixed points of asymptotically nonexpansive mappings, J. Math. Anal. Appl. 67 (1979) 274-276.

[23] Schu J.: Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull. Math. Soc. 43 (1991) 153-159.

[24] Semple C.: Phylogenetics. Oxford Lecture Series in Mathematics and Its Application. Oxford University Press, Oxford (2003).

[25] Shahzad N.: Approximating fixed points of non-self nonexpansive mappings in Banach spaces, Nonlinear Anal.61 (2005) 1031-1039.

[26] Sun Z.H.: Strong convergence of an implicit iteration process for a finite family of asymptrotically quasi-nonexpansive mappings. J. Math. Anal. Appl. 286 (2003) 351-358.

[27] Takahashi W., Kim G.E.: Strong convergence of approximants to fixed points of nonexpansive nonself-mappings, Nonlinear Anal.32 (1998) 447-454.

[28] Tan K.K., Xu H.K.: Approximating fixed points of nonexpansive mapping by the Ishikawa iteration process, J. Math. Anal. Appl. 178 (1993) 301-308.

[29] Tan K.K., Xu H.K.: Fixed point iteration process for asymptotically nonexpansive mappings. Proc. Amer. Math. Xoc. 122(3) (1994) 733-739.

[30] Thianwan S.: Common fixe point of new iteration for two asymptotically nonnexpansive nonself-mapping in a Banach space. Feb. 15 (2009) 688-695.

[31] Wang L.: Strong and weak convergence theorems for common fixed points of nonself asymptotically nonexpansive mappings, J. Math. Anal. Appl. 323 (2006) 550-557.

[32] Xu H.K., Yin X.M.: Strong convergence theorems for nonexpansive nonself-mappings, Nonlinear Anal. 242 (1995) 23-228.

[33] Zhao-hong Sun: Strong convergence of an implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings. J. Math. Anal. Appl. 286 (2003) 351-358.

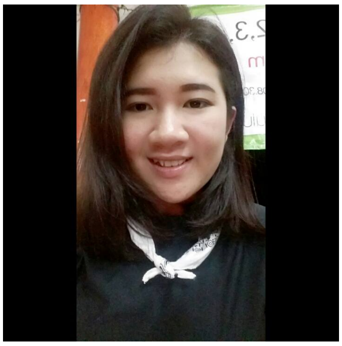
BIOGRAPHY

Name Surname	Ruethaita Yeampun
Date of Birth	2 June 1994
Education Background	Junior High School from Buddhachinnarajpittaya School in 2009, Muang, Phitsanulok, Thailand Senior High School from Buddhachinnarajpittaya School in 2012, Muang, Phitsanulok, Thailand
Address	House No. 611/73 Village No. 8 Wangthong sub-district, Wangthong district, Phitsanulok province 56000
Telephone number	0812849398
E-mail	pumisslovemilk@hotmail.com

BIOGRAPHY

Name Surname	Wilipron Witoon
Date of Birth	6 October 1994
Education Background	Junior High School from Chiang Saen Witthayakom School in 2009, Chiang Saen, Chiang Rai, Thailand Senior High School from Chiang Saen Witthayakom School in 2012, Chiang Saen, Chiang Rai, Thailand
Address	House No. 673 Village No. 3 Waing sub-district, Chiang Saen district, Chiang Rai province 57150
Telephone number	0920319406
E-mail	popza@hotmail.com

BIOGRAPHY



Name Surname	Intira Kapang
Date of Birth	11 December 1994
Education Background	Junior High School from Prakhonchaiwitthaya School in 2009, Prakonchai, Buriram, Thailand Senior High School from Prakhonchaipitthaya School in 2012, Prakonchai, Buriram, Thailand
Address	House No. 28 Village No. 6 sub-district, Prakonchai district, Buriram province 31140
Telephone number	0874428275
E-mail	kratai@hotmail.com